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Schur Algebras, Combinatorics, and Cohomology

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This thesis is dedicated to my mother, for all her love and support,
and for providing me with the space to find my own way.

Contents

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Declaration

Except where stated to the contrary this thesis is my own work.

Summary

Let $G = GL_n(k)$, the group of all invertible $n \times n$ matrices over an infinite field k. In this thesis we explore the cohomological relationship between a Schur algebra S(G) for G and the subalgebra S(B) corresponding to a Borel subgroup B of G. Our main motivation is the question of whether there is an analogue of the Kempf Vanishing Theorem in this setting.

We place our study in a more general framework, defining subalgebras $S(\Omega,\Gamma)$ of S(G) associated with certain intersections of parabolic subgroups of G, and investigate the connection between $S(\Omega,\Gamma)$ and the subalgebra $S(\Omega,\emptyset)$. We define modules for $S(\Omega,\Gamma)$ which serve as analogues for the Weyl modules for S(G). We produce bases for these Weyl modules and thereby show that $S(\Omega,\Gamma)$ is a quasi-hereditary algebra.

We find two-step projective presentations for the Weyl modules over subalgebras $S(\Omega,\Gamma')$ of $S(\Omega,\Gamma)$, and in special cases find projective resolutions. We use these to prove results which provide partial information on the existence of an analogue for the Kempf Vanishing Theorem, and on related questions.

We derive a character formula for the Weyl modules which can be regarded as an extension of the Jacobi-Trudi identity for Schur functions.

The methods used in this thesis are in the main elementary, with a heavy reliance on direct combinatorial arguments.

I: Introduction

Let G be a connected reductive algebraic group over an algebraically closed field k. There is a striking relationship between the rational representation theory of G and that of a Borel subgroup B. Let T be a maximal torus of G contained in B. B determines a set of positive roots in the root system of G with respect to T, and hence gives rise to a dominance ordering on the set of weights (=characters) of T. So that we deal with dominant rather than anti-dominant weights we will use the dominance order obtained from the opposite Borel subgroup B^-. The simple rational B-modules are in one to one correspondence with the weights of T, and if $k(\lambda)$ is the simple module associated with the weight λ, then the induced module $\mathrm{Ind}_B^G k(\lambda)$ is non-zero if and only if λ is a dominant weight. Moreover, in the latter case $\mathrm{Ind}_B^G k(\lambda)$ has a simple socle, and these socles form a complete set of simple rational G-modules.

Each rational B-module V determines a sheaf $\mathcal{L}(V)$ of $\mathcal{O}_{G/B}$-modules (where $\mathcal{O}_{G/B}$ denotes the structure sheaf of the quotient variety G/B), whose sheaf cohomology coincides with the values on V of the right derived functors of induction:

$$H^i(G/B, \mathcal{L}(V)) \cong R^i \mathrm{Ind}_B^G(V) \quad \forall i \geq 0.$$

The Kempf Vanishing Theorem [K] asserts that if λ is a dominant weight then

$$H^i(G/B, \mathcal{L}(k(\lambda))) = 0 \quad \forall i > 0;$$

in other words $k(\lambda)$ is right Ind_B^G-acyclic (cf. [CPSK]) [1]. This theorem is fundamental for the cohomology theory of rational G-modules. It can be used to show that if V is a rational G-module then

$$R^i \mathrm{Ind}_B^G V \cong \begin{cases} V & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases},$$

[1] In [K] and [CPSK] G is assumed to be semisimple, but the passage to reductive G is straightforward.

so that Ind_B^G takes any B-injective resolution of V to a G-injective resolution of V (see the proof of [CPSK; (2.1)]). It follows that $\mathrm{Ext}^i_G(V',V) \cong \mathrm{Ext}^i_B(V',V)$ for all rational G-modules V,V' and all $i \geq 0$. The Kempf Vanishing Theorem is also used in [CPSK] to prove that for any dominant weights λ and μ,

$$H^i(G, \mathrm{Ind}_B^G k(\lambda) \otimes \mathrm{Ind}_B^G k(\mu)) = 0 \quad \forall\, i > 0.$$

This fact is invoked in the production of good filtrations (see [Do1]), and thereby in the description of the category of rational G-modules as a highest weight category (see [CPS1], [CPS2]). Kempf's original proof is rather long and technical, and shorter proofs have been given (see for example [An] and [H]). None of these are completely representation-theoretic: they rely on techniques from sheaf cohomology theory. See however the appendix to [Do2] .

Taking the foregoing as motivation we now concentrate on the case $G = GL_n(k)$. For $f \in \mathbb{N}$ let $S = S(G)$ be the Schur algebra associated with n and f (cf. [G1]), so that mod$S$ is the category of homogeneous polynomial representations of degree f of G. The cohomology theory of S has received attention in recent years. It was proved independently by Akin and Buchsbaum in [AB2], and by Donkin in [Do3], that S has finite global dimension. In [AB2] this is accomplished by giving an inductive procedure for the construction of finite projective resolutions of Weyl modules. In [Do3] it is proved for a more general class of algebras (analogues of the Schur algebras for arbitrary reductive groups), using the machinery of good filtrations. These generalized Schur algebras (more precisely algebras Morita equivalent to them) have also featured in the work of Cline, Parshall and Scott, as examples of quasi-hereditary algebras (see [CPS1]). We remark that although the Kempf Vanishing Theorem is used in proving that these algebras are quasi-hereditary, this is not necessary for S: a short direct proof is given in [P]. [2]

To each closed subgroup H of G there corresponds a subalgebra S(H) of S(G). This thesis started as an attempt to see how far the relationship between G and a Borel

[2] There is also a heredity chain for S implicit in the 'fundamental filtration' of [DEP; §3]: take the part of their filtration of the polynomial coordinate ring which is homogeneous of degree f, then apply contravariant duality.

subgroup B was reflected in the relationship between S(G) and the subalgebra S(B). For computational simplicity we take $B = B^-$, the Borel subgroup of all lower triangular matrices in G, and denote by B^+ the opposite Borel subgroup consisting of all upper triangular matrices. It is known that there is a correspondence between simple S(G)-modules and induced simple S(B)-modules which parallels the case for the rational categories. *We ask further whether the following statements are true*

(A) The simple S(B)-module $k(\lambda)$ is $\mathrm{Ind}_{S(B)}^{S(G)}$-acyclic for all dominant weights λ.

(B) The restriction functor mod S(G) → mod S(B) preserves Ext groups.

One of the aims in [AB1] and [AB2] is the construction of explicit (finite) S(G)-projective resolutions of Weyl modules. If (A) holds we have an alternative approach to this problem: for dominant λ find $S(B^+)$-projective resolutions of the simple modules $k(\lambda)$, then apply the functor $S(G)\otimes_{S(B^+)}$. The problem of constructing such resolutions is considered in [Sa].

The study of (A) and (B) led naturally to a study of the subalgebras of S associated with the parabolic subgroups of G containing B. The construction of analogues of the Weyl modules for these parabolic Schur algebras suggested the consideration of a more general type of subalgebra of S, obtained in the following way: take a parabolic subgroup P containing B, and a second parabolic subgroup Q containing the opposite Borel subgroup B^+. P and Q can be specified by giving a pair (Ω,Γ) of subsets of the set Δ of simple roots (see II.1.2). We form the subalgebra $S(\Omega,\Gamma) = S(P\cap Q)$ of S associated with their intersection. Initially we were interested only in the cases (Δ,Ω), $(\emptyset,\Omega)$, (Ω,Δ) and $(\Omega,\emptyset)$. $S(\Omega,\Gamma)$ was introduced for notational convenience and to simplify proofs that were originally split into separate cases. It was noticed however that almost all the results and proofs could be adapted with only minor changes for general (Ω,Γ). The only place where significant extra work was required was in proving that $S(\Omega,\Gamma)$ is a quasi-hereditary algebra. In view of the current interest in quasi-hereditary algebras (and the conceptual simplification obtained by working within a uniform framework) it seemed worthwhile to treat the general case.

In chapter II we give a treatment of the algebra $S(\Omega,\Gamma)$. We define Weyl

modules for $S(\Omega,\Gamma)$ by inducing simple modules from the subalgebra $S(\Omega,\emptyset)$, and show that they share many of the properties of the classical Weyl modules. We use these Weyl modules to classify the simple modules in $\mathrm{mod}S(\Omega,\Gamma)$, and we prove that $S(\Omega,\Gamma)$ is a quasi-hereditary algebra by exhibiting an explicit heredity chain, much as in [P]. This implies in particular that $S(\Omega,\Gamma)$ has finite global dimension (see [CPS1], [DR]). Put $\Psi = \Omega \cap \Gamma$. *Generalizing (A) and (B) we ask whether the following statements are true*

(A') The simple $S(\Omega,\emptyset)$-module $k(\lambda)$ is $\mathrm{Ind}_{S(\Omega,\emptyset)}^{S(\Omega,\Gamma)}$-acyclic for all Ψ-dominant weights λ. (See I.1.4 for the definition of Ψ-dominance.)

(B') If $\Omega \supseteq \Gamma$ the restriction functor $\mathrm{mod}\, S(\Omega,\Gamma) \to \mathrm{mod}\, S(\Omega,\emptyset)$ preserves Ext groups. (The condition $\Omega \supseteq \Gamma$ is necessary – see III.3.5.)

Using the quasi-hereditary nature of $S(\Omega,\Gamma)$ we show that (A') is equivalent to the vanishing of $\mathrm{Ext}^i_{S(\Omega,\emptyset)}(V(\Omega,\Gamma,\lambda), k(\mu))$ for all Ψ-dominant weights λ, μ and all $i > 0$, where $V(\Omega,\Gamma,\lambda)$ denotes the Weyl module with highest weight λ. This observation leads us in chapter III to the investigation of the Weyl modules for $S(\Omega,\Gamma)$ as modules for the subalgebra $S(\Omega,\emptyset)$. We show that to establish (B') it is enough to show that when $\Omega \supseteq \Gamma$, $\mathrm{Tor}_i^{S(\Omega,\emptyset)}(S(\Omega,\Gamma), V(\Omega,\Gamma,\lambda))$ is zero for all Ψ-dominant weights λ and all $i > 0$. *Thus (A') and (B') are true if for all Ψ-dominant weights λ the following statement is true*

(C)
$$\begin{cases} \mathrm{Ext}^i_{S(\Omega,\emptyset)}(V(\Omega,\Gamma,\lambda), k(\mu)) = 0 \quad \forall\, i > 0, \forall\, \Psi\text{-dominant weights } \mu. \\ \mathrm{Tor}_i^{S(\Omega,\emptyset)}(S(\Omega,\Gamma), V(\Omega,\Gamma,\lambda)) = 0 \quad \forall\, i > 0. \end{cases}$$

Let λ be a Ψ-dominant weight. Using a couple of combinatorial lemmas on the multiplication in $S(\Omega,\Gamma)$ we produce a two-step $S(\Omega,\emptyset)$-projective presentation of $V(\Omega,\Gamma,\lambda)$, which enables us to prove a first case of (A'), namely that

$$R^1\mathrm{Ind}_{S(\Omega,\emptyset)}^{S(\Omega,\Gamma)} k(\lambda) = 0.$$

In producing the presentation mentioned above we introduce submodules $M_\alpha = M_\alpha(\Omega,\Gamma,\lambda)$ of $S(\Omega,\Gamma)^\lambda$. We go on to construct a projective resolution of the module $S(\Omega,\Gamma)^\lambda / M_\alpha$, which we use to prove (C) when λ has a certain restricted form. The relevant condition is

$$\#\{\alpha_b \in \Psi \,/\, \lambda_b \geq \lambda_{b+1} \geq 1\} \leq 1.$$

(See I.1.3 for the definition of α_b.) If $|\Psi| \leq 1$ all Ψ-dominant weights satisfy this condition, so (A') and (B') hold in this case. In particular (A) and (B) hold when $n = 2$. An interesting feature of the techniques used in chapter III is that they apply uniformly to both $S(\Omega,\Gamma)$ and $S(\Omega,\emptyset)$.

In chapter III we only get to grips fully with Weyl modules $V(\Omega,\Gamma,\lambda)$ such that the kernel of the projection $S(\Omega,\emptyset)^\lambda \to V(\Omega,\Gamma,\lambda)$ 'involves no more than a single root'. In chapter IV we give some partial results concerning cases where this kernel involves more than one root. In §1 we show that (C) holds for Ψ-hook weights, the appropriate generalization to our setting of hook partitions. We do this by constructing explicit projective resolutions. In §2 we restrict ourselves to characteristic zero and $\Omega = \Delta$, and show that the resolutions of §1 are special cases of a complex which exists for all λ, and which is related to the Bernstein-Gelfand-Gelfand resolution of the simple $\mathfrak{sl}_n(k)$-module of highest weight λ. When $n \leq 3$ we show that this complex is exact, and so establish (A) and (B) for characteristic zero and $n \leq 3$. In §3 we derive a character formula for $V(\Omega,\Gamma,\lambda)$ which generalizes the Jacobi-Trudi identity for Schur functions. This formula relates the character of $V(\Omega,\Gamma,\lambda)$ to the characters of the projective modules $S(\Omega,\Gamma')^\mu$ for subsets $\Gamma' \subseteq \Psi$. When $\Omega = \Delta$ the formula shows that the Euler characteristic of the complex of §2 is zero. In §4 we show that (A') holds when $n = 3$.

For the most part we will not involve ourselves with the general representation theory of algebraic groups, and our methods will be mainly elementary. We use the basic representation theory of finite-dimensional algebras, some homological algebra, and various combinatorial arguments. From time to time we will point out connections between our situation and that for algebraic groups. For this purpose we fix the following notation: as above B^- and B^+ denote the Borel subgroups of

$G = GL_n(k)$ consisting respectively of all lower and all upper triangular matrices, and $T = B^- \cap B^+$ denotes the maximal torus of diagonal matrices. We work over an arbitrary infinite field k.

§1 Combinatorial Preliminaries

In this section we introduce the basic combinatorial ideas and notation which will be used throughout the work.

I.1.1 Weights

$\Lambda(n) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (n copies) is the set of ***weights*** [3]. We define orders on $\Lambda(n)$:

$\lambda \subseteq \mu$ if $\lambda_a \leq \mu_a \ \forall\, a \in \underline{n}$ (λ is a ***subweight*** of μ).

$\lambda \trianglelefteq \mu$ if $\lambda_1 + \cdots + \lambda_a \leq \mu_1 + \cdots + \mu_a \ \forall\, a \in \underline{n}$ (the ***dominance*** order).

$\lambda <_{lex} \mu$ if $\lambda \neq \mu$ and the first non-zero difference $\lambda_a - \mu_a$ is negative (the ***lexicographic*** order).

We put $\lambda \lhd \mu$ if $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$; $\lambda \subset \mu$ if $\lambda \subseteq \mu$ and $\lambda \neq \mu$; and $\lambda \leq_{lex} \mu$ if $\lambda <_{lex} \mu$ or $\lambda = \mu$. $\subseteq$ and $\trianglelefteq$ are partial orders, $\leq_{lex}$ a total order. We have

$$\lambda \subseteq \mu \Rightarrow \lambda \trianglelefteq \mu \Rightarrow \lambda \leq_{lex} \mu.$$

The lexicographic order behaves well with respect to addition, in the sense that

$$\lambda \leq_{lex} \mu\,,\ \lambda' \leq_{lex} \mu' \Rightarrow \lambda+\lambda' \leq_{lex} \mu+\mu',$$

with equality on the right iff we have equality in both cases on the left.

The number $|\lambda| = \lambda_1 + \cdots + \lambda_n$ is the ***degree*** of the weight λ. We will call λ a ***polynomial*** weight if $\lambda \supseteq 0$, and for $f \in \mathbb{N}_0$ we define

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[3] $\Lambda(n)$ identifies with the set of weights of T, $(\lambda_1, \cdots, \lambda_n)$ corresponding to the map $T \to k$ which sends the matrix with diagonal entries $t_1, t_2, \cdots, t_n$ to $(t_1)^{\lambda_1} \cdots (t_n)^{\lambda_n}$. (cf. [G1; (3.2)].)

$$\Lambda(n,f) = \{\lambda \in \Lambda(n) \,/\, \lambda \geq 0 \text{ and } |\lambda| = f\},$$

the set of ***polynomial weights of degree f***. λ is ***dominant*** if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (see also I.1.4).

I.1.2 Indices

Let X be a finite subset of $\mathbb{N}$. We will denote by $\mathbf{I(n,X)}$ the set of all maps $X \to \underline{n}$. The elements of I(n,X) will be referred to as ***indices***, and if $i \in I(n,X)$, $x \in X$ we will write i_x for $i(x)$. If Y is a subset of X we will sometimes refer to the values i_y for $y \in Y$ as the ***entries*** of i in Y. Indices will be used to parameterize bases for certain modules.

Any $i \in I(n,X)$ has an associated weight $\lambda = \mathbf{wt(i)} \in \Lambda(n, |X|)$, defined by

$$\lambda_a = \#\{x \in X \,/\, i_x = a\} \quad (a \in \underline{n}).$$

If $i,j \in I(n,X)$ we will write

$$i \leq j \quad \text{if } i_x \leq j_x \ \forall\, x \in X.$$

We denote by $\mathbf{P(X)}$ the group of all permutations of X. P(X) acts on the right of I(n,X) by

$$(i\pi)_x = i_{\pi(x)} \quad i \in I(n,X),\ \pi \in P(X),\ x \in X.$$

$i,j \in I(n,X)$ lie in the same P(X)-orbit iff wt(i) = wt(j), in which case we write $i \sim j$. For each $\lambda \in \Lambda(n,|X|)$ we define the ***canonical index*** of weight λ to be the unique index $i \in I(n,X)$ of weight λ which is non-decreasing $(x,y \in X,\ x \leq y \Rightarrow i_x \leq i_y)$. The various canonical indices form a set of orbit representatives for the P(X)-action on X.

P(X) also acts on the set $I(n,X) \times I(n,X)$ by $(i,j)\pi = (i\pi, j\pi)$, and we write $(i,j) \sim (i',j')$ if (i,j) and (i',j') lie in the same P(X)-orbit. Notice that the partial order $\leq$ is compatible with this P(X)-action.

Each partition $\{X_1, X_2, \ldots, X_r\}$ of X determines a ***Young subgroup***

$P(X_1) \times \cdots \times P(X_r) \leq P(X)$.

We adopt the following notation from [G2; §2]: for each collection of indices $i(1), i(2), \cdots, i(r) \in I(n,X)$, and each collection $a_1, \cdots, a_r$ of elements of $\underline{n}$, we define a subset of X by:

$$R_{a_1,a_2,\cdots,a_r}(i(1), i(2), \cdots, i(r)) = \{x \in X \mid i(\rho)_x = a_\rho \ \forall \rho \in \underline{r}\}.$$

We put

$$P_{i(1),\cdots,i(r)} = \prod_{a_1,\cdots,a_r} P(R_{a_1,\cdots,a_r}(i(1),\cdots,i(r))),$$

the Young subgroup of P(X) corresponding to the partition

$$\{ R_{a_1,a_2,\cdots,a_r}(i(1), i(2), \cdots, i(r)) \}_{a_1,\cdots,a_r \in \underline{n}}$$

If $X = \underline{f}$ we will write $I(n,f)$ for $I(n,\underline{f})$ and $P(f)$ for $P(\underline{f})$.

I.1.3 Roots and the Weyl Group [4]

For $a \in \underline{n}$ define $\varepsilon_a \in \Lambda(n)$ by $(\varepsilon_a)_b = \delta_{a,b}$ $(b \in \underline{n})$. Then $\varepsilon_1, \cdots, \varepsilon_n \in \Lambda(n) \subseteq \mathbb{R}^n$ are the standard basis vectors of $\mathbb{R}^n$. Consider $\mathbb{R}^n$ as a Euclidean space with the usual inner product given by $(\varepsilon_a, \varepsilon_b) = \delta_{a,b}$. Then

$$\Phi = \{ \varepsilon_a - \varepsilon_b \mid a,b \in \underline{n}, a \neq b \}$$

is a root system of type A_{n-1} in $\mathbb{R}\Phi$, and

$$\Delta = \{ \alpha_a = \varepsilon_a - \varepsilon_{a+1} \mid a \in \underline{n-1} \}$$

a set of simple roots. We denote by Φ^+ the set of positive roots, i.e. the subset of Φ

[4] A suitable reference for this subsection is the appendix to [St].

consisting of those roots which are expressible as non-negative integer combinations of elements from Δ [5]. Denote by W the ***Weyl group*** of Φ, i.e. the group generated by the reflections:

$$s_\alpha : x \mapsto x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\cdot\alpha \qquad \alpha\in\Phi,\ x\in\mathbb{R}^n.$$

If $\alpha\in\Delta$, s_α will be called a simple reflection. The ***length*** $l(w)$ of $w\in W$ is the length of a minimal expression for w as a product of simple reflections. We record for future use the following fact: if $w\in W$ and $\alpha\in\Delta$, then $l(s_\alpha w) = l(w) + 1$ iff $w^{-1}\alpha\in\Phi^+$.

The action of P(n) on $\mathbb{R}^n$ by place permutations:

$$\pi(x_1, \cdots, x_n) = (x_{\pi^{-1}(1)}, \cdots, x_{\pi^{-1}(n)}) \qquad (x_1, \cdots, x_n)\in\mathbb{R}^n,\ \pi\in P(n),$$

identifies P(n) with W. The reflection s_{α_a} corresponds to the transposition (a a+1).

We will also need the so-called ***dot-action*** of W on $\mathbb{R}^n$ given by

$$w\circ x = w(x+\delta)-\delta \qquad w\in W,\ x\in\mathbb{R}^n,$$

where δ is the weight $(n-1, n-2, \cdots, 0)$. Notice that if $\alpha\in\Phi$, (δ,α) is the ***height*** of α, i.e. the sum of the coefficients when α is expressed as a linear combination of simple roots. In particular $(\delta,\alpha) > 0$ if $\alpha\in\Phi^+$, and $(\delta,\alpha) = 1$ if $\alpha\in\Delta$.

W acts on I(n,X) on the left by $(wi)_x = w(i_x)$ $(x\in X)$, and this action commutes with that of P(X) on the right.

I.1.4 Parabolic subgroups of W

If $\Psi\subseteq\Delta$, denote by W_Ψ the ***parabolic*** subgroup of W generated by the reflections s_α for $\alpha\in\Psi$. This is the Young subgroup of P(n) corresponding to the partition:

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[5] Φ is the root system of G with respect to the torus T. The positive roots Φ^+ corresponding to the simple roots Δ are those associated with the Borel subgroup B^+ of upper triangular matrices. The dominance order on $\Lambda(n)$ as defined in I.1.1 is that defined by this set of simple roots.

$$\{1, 2, \cdots, a_1\} \ , \ \{a_1+1, \cdots, a_2\} \ , \cdots , \ \{a_t+1, \cdots, n\}$$

of $\underline{n}$, where $\{\alpha_{a_1}, \cdots, \alpha_{a_t}\} = \Delta \backslash \Psi$. We will refer to the W_Ψ-orbits of $\underline{n}$ as *Ψ-blocks*, and write $a \sim_\Psi b$ if a and b are in the same Ψ-block.

If $i,j \in I(n,X)$ we write

$$i \sim_\Psi j \text{ if } i_x \sim_\Psi j_x \ \forall x \in X.$$

The relation $\sim_\Psi$ is compatible with the action of $P(X)$ on $I(n,X) \times I(n,X)$.

Dominance

A weight $\lambda \in \Lambda(n)$ is *Ψ-dominant* if $(\lambda, \alpha) \geq 0 \quad \forall \alpha \in \Psi$. This is equivalent to the condition $w\lambda \trianglelefteq \lambda \ \forall w \in W_\Psi$, and to $\lambda_a \geq \lambda_b \ \forall a,b \in \underline{n}$ with $a \sim_\Psi b$ and $a \leq b$. Each weight λ is W_Ψ-conjugate to exactly one Ψ-dominant weight.

Noting that if $\lambda, \mu \in \Lambda(n,f)$, then $\lambda \trianglelefteq \mu$ iff $\mu - \lambda$ is a sum of simple roots, we define a partial order on $\Lambda(n,f)$ by

$$\lambda \trianglelefteq_\Psi \mu \quad \text{if } \mu - \lambda \text{ is a sum of elements taken from } \Psi.$$

I.1.5 Tableaux and Standardness

For $\lambda \in \Lambda(n,f)$ let $[\lambda]$ denote the *shape* of λ, i.e. the subset

$$\{(a,b) \in \mathbb{N}^2 \ / \ a \in \underline{n}, \ b \in \{1, 2, \cdots, \lambda_a\}\}$$

of $\mathbb{N}^2$. A *λ-tableau* is a map $[\lambda] \to \mathbb{N}$. We define a particular λ-tableau $T^\lambda = T : [\lambda] \to \underline{f}$, by requiring that T be order-preserving when $\mathbb{N}^2$ is ordered lexicographically. Since T is bijective, any other tableau factors uniquely through T, so T sets up a bijection between $I(n,f)$ and the set of λ-tableaux with values in $\underline{n}$. If $i \in I(n,f)$ we write T_i for the composite $[\lambda] \xrightarrow{T} \underline{f} \xrightarrow{i} \underline{n}$.

We will follow the usual convention when drawing $[\lambda]$ of applying a 90^0 clockwise rotation to the standard representation of $\mathbb{N}^2$, so that the first coordinate

increases from top to bottom and the second from left to right. We will represent a tableau graphically by writing its values into the appropriate places in the diagram of $[\lambda]$.

Example

If n=4, f=17, λ=(6,4,5,2) then $\mathbb{T}$ is depicted by:

1	2	3	4	5	6
7	8	9	10		
11	12	13	14	15	
16	17				

and if l is the canonical index of weight λ , $\mathbb{T}_l$ is depicted by:

1	1	1	1	1	1
2	2	2	2		
3	3	3	3	3	
4	4				

For $\Psi \subseteq \Delta$, $a \in \mathbb{N}$ and Ξ a Ψ-block of $\underline{n}$ we define via $\mathbb{T}$ certain subsets of $\underline{f}$:

$$\mathbf{R}_a = \mathbb{T}([\lambda] \cap (\{a\} \times \mathbb{N})) \qquad \text{(the } \textit{rows}\text{)}$$

$$\mathbf{C}_a = \mathbb{T}([\lambda] \cap (\mathbb{N} \times \{a\})) \qquad \text{(the } \textit{columns}\text{)}$$

$$\mathbf{C}_{\Xi,a} = \mathbb{T}([\lambda] \cap (\Xi \times \{a\})) \qquad \text{(the } \textit{Ψ-columns}\text{)}.$$

Each of the collections of subsets $\{\mathbf{R}_a\}$, $\{\mathbf{C}_a\}$, $\{\mathbf{C}_{\Xi,a}\}$ is a partition of $\underline{f}$. We will denote the Young subgroup of P(f) corresponding to the partition $\{\mathbf{C}_{\Xi,a}\}$ by $\mathbf{C}(\lambda,\Psi)$. These definitions depend upon the choice of the weight λ, and if we want to stress this, we will talk of λ-rows etc..

If $i \in I(n,X)$ and $Y \subseteq X$ we will say that i is ***standard*** (resp. ***semi-standard***) on Y if i is strictly increasing (resp. non-decreasing) when restricted to Y. If Π is a set of subsets of X we will say that i is standard (resp. semi-standard) on Π if it is so on every $Y \in \Pi$. Similarly we define ***reverse standard*** (resp. ***reverse semi-standard***) by

requiring that i be strictly decreasing (resp. non-increasing) on the relevant subsets of X. If X=f we will call i ***row standard***, ***column standard***, ***Ψ-column standard*** etc. if it is so on the partitions of f given by the rows, columns, Ψ-columns resp..

If t: $[\lambda] \to \underline{n}$ is a λ-tableau we define notions of standardness for t in terms of the index $t\cdot(T^{\lambda})^{-1} \in I(n,f)$. We choose to make our definitions in this somewhat contrary manner because for the most part we will be dealing with indices rather than tableaux, and we usually want rows, columns etc. to refer to subsets of f not subsets of $[\lambda]$.

I.1.6 Lemma

Suppose $i \in I(n,f)$ is Ψ-column standard.

(i) Suppose λ is Ψ-dominant, and let j be the index obtained from i by permuting the entries in each row so that they become semi-standard. Then j is Ψ-column standard.

(ii) Suppose λ is ***Ψ-anti-dominant***, (i.e. $\lambda_a \leq \lambda_b$ whenever $a \leq b$ and a, b are in the same Ψ-block), and let j be the index obtained from i by permuting the entries in each row so that they become reverse semi-standard. Then j is Ψ-column standard.

Proof

(i) Since the problem is local to each Ψ-block we may assume that Ψ=Δ, and then the result is [De; Lemme 1] [6].

(ii) This follows either by modifying the proof for (i) or can be deduced from (i) in the following manner. We may again assume that Ψ=Δ. Let w be the longest element of W, i.e. the permutation which reverses the order of the elements in n, and let ω be the composite

$$\underline{f} \xrightarrow[(T^{\lambda})^{-1}]{} [\lambda] \to [w\lambda] \xrightarrow[T^{w\lambda}]{} \underline{f},$$

where the middle map is given by $(a,b) \mapsto (wa,b)$. Consider the map which takes an index m to $wm\omega^{-1}$. This takes λ-column standard indices to wλ-column standard indices and takes reverse row semi-standard indices to row semi-standard indices. Thus the index $wi\omega^{-1}$ is column standard for the dominant weight wλ, so by (i) the

6 To pass from our situation to that of [De] rows and columns should be interchanged.

index obtained by reordering its rows so that they become row semi-standard is column standard for $w\lambda$. However this index is $wj\omega^{-1}$, and so j is λ-column standard. □

§2 The Schur Algebra

The Schur algebra was first investigated by Schur in his dissertation [S] in which he classified the polynomial representations of $GL_n(\mathbb{C})$. In this section we recall the definition of the Schur algebra as given in [G1].

Let $M = M_n(k)$ be the affine algebraic monoid of all $n \times n$ matrices over k, and $G = GL_n(k)$ the group of all invertible matrices in M. Let $c_{a,b}$ $(a,b \in \mathbb{N})$ be the map which sends a matrix $m \in M$ to its (a,b)-coordinate $m_{a,b}$. The coordinate ring k[M] of M is a polynomial ring in the $c_{a,b}$. For $i,j \in I(n,f)$ put

$$c_{i,j} = \prod_{\varphi \in \underline{f}} c_{i_\varphi, j_\varphi} \quad \text{(a monomial of total degree f)}.$$

If also $i',j' \in I(n,f)$ then $c_{i,j} = c_{i',j'}$ iff $(i,j) \sim (i',j')$. The set

$$\{c_{i,j} \,/\, f \in \mathbb{N}_0 \,,\, i,j \in I(n,f)\}$$

is a basis for k[M].

The monoid structure of M endows k[M] with the structure of a k-coalgebra, with comultiplication

$$\nabla : k[M] \to k[M] \otimes k[M]$$
$$\nabla(c_{i,j}) = \sum_{j \in I(n,f)} c_{i,j} \otimes c_{j,l} \quad \text{if } i,l \in I(n,f),$$

and counit

$$\varepsilon : k[M] \to k$$
$$\varepsilon(c_{i,j}) = \delta_{i,j}.$$

The category of k[M]-comodules is isomorphic to the category of rational representations of M, and to the category of polynomial representations of G. (See [G1; §1].)

For $f \in \mathbb{N}_0$ denote by $A = A_f$ the subcoalgebra of k[M] consisting of those polynomials in the $c_{a,b}$ which are homogeneous of degree f. A_f has basis:

(a) $$\{c_{i,j} \mid i,j \in I(n,f)\}.$$

We have

$$k[M] = \bigoplus_{f \in \mathbb{N}_0} A_f,$$

and this decomposition essentially reduces the representation theory of k[M] to that of the finite dimensional coalgebras A_f, and hence to that of their dual algebras. Thus, let $S = S_f$ be the dual algebra of A_f. This is the ***Schur Algebra*** ([G1; (2.3)]). We let

(b) $$\{\xi_{i,j} \mid i,j \in I(n,f)\}$$

be the basis of S dual to (a). We see that $\dim S = \dim A = \binom{n^2+f-1}{f}$. If $i \in I(n,f)$ has weight μ we will write ξ_μ for $\xi_{i,i}$ and c_μ for $c_{i,i}$.

Using the duality of the bases (a), (b) and the comultiplication formula given above, we deduce the multiplication formula of [G1; (2.3)]:

(c) $$\xi_{i,j} \cdot \xi_{l,m} = \sum_{(q,r)} Z(i,j,l,m,q,r) \cdot \xi_{q,r},$$

where

$$Z(i,j,l,m,q,r) = \#\{ s \in I(n,f) \mid (i,j) \sim (q,s) \text{ and } (l,m) \sim (s,r)\},$$

and the summation is over a set of representatives (q,r) of the P(f)-orbits of $I(n,f) \times I(n,f)$. In particular $\xi_\mu \cdot \xi_{i,j}$ is $\xi_{i,j}$ if $wt(i) = \mu$, and zero otherwise.

A Relative Order on Indices

For $i,j,l \in I(n,f)$, define

$i \sim_l j$ if $i|_{R_a(l)} \sim j|_{R_a(l)}$ $\forall a \in \underline{n}$.

$i <_l j$ if $i \neq_l j$, and if the first non-zero difference $wt(i|_{R_a(l)}) - wt(j|_{R_a(l)})$ is $>_{lex} 0$.

We will write $i \leq_l j$ if $i <_l j$ or $i = j$. We have $i \sim_l j$ iff $\xi_{i,l} = \xi_{j,l}$. Note that $\leq_l$ is only a partial order on I(n,f), although it induces a total order on the set of equivalence classes of the relation $\sim_l$ and hence on the set of basis elements $\xi_{q,r}$ with $r \sim l$.

II: Weyl Modules

In this chapter we define for each pair of subsets Ω,Γ of the set of simple roots Δ a subalgebra $S(\Omega,\Gamma)$ of the Schur algebra S. $S(\Omega,\Gamma)$ is (in a sense to be made precise below) the 'image' in S of a certain subgroup of $GL_n(k)$. We define modules $V(\Omega,\Gamma,\lambda)$ and $D(\Omega,\Gamma,\lambda)$ for $S(\Omega,\Gamma)$ which will be referred to as Weyl and Schur modules respectively. These modules play the part in the representation theory of $S(\Omega,\Gamma)$ that the usual Weyl and Schur modules play in that of the Schur algebra. We show that $S(\Omega,\Gamma)$ is a quasi-hereditary algebra, and hence that its module category is a highest weight category in the sense of [CPS2]. We show that $V(\Omega,\Gamma,\lambda)$ and $D(\Omega,\Gamma,\lambda)$ are the modules $V(\lambda)$ and $A(\lambda)$ of [CPS2] for this highest weight category. When $\Omega=\Gamma=\Delta$ we recover the classical case.

We will be considering the algebra $S(\Omega,\emptyset)$ and its relationship to $S(\Omega,\Gamma)$. For each result in this context there is a corresponding result about $S(\emptyset,\Gamma)$ which we will not write down but which we may use subsequently. The formulation of these transposed results is a formal exercise which we leave to the reader.

We warn the reader that this chapter is not self-contained: at the start of §3 we quote a result on the dimension of $V(\Omega,\Gamma,\lambda)$ whose proof we defer until the next chapter. We do this because the required result will be obtained as part of a general framework which is conceptually distinct from the ideas of this chapter.

§1 The Algebra $S(\Omega,\Gamma)$

Firstly we make some general comments about sub-monoids of M and subalgebras of S. If X is a subset of M, denote by $k[X]$ the quotient of $k[M]$ by the ideal of functions which vanish on X, i.e. $k[X] = k[\overline{X}]$ is the coordinate ring of the Zariski closure $\overline{X}$ of X. Denote by $A(X)$ the image of $A = A_f$ under the canonical map $k[M] \to k[X]$, and by $S(X)$ its dual space. The canonical map $S(X) \to S$ identifies $S(X)$ with $k\cdot e(X)$, where e is the evaluation map

$$e: M \to S$$
$$e(m)(c) = c(m) \quad m \in M, c \in A.$$

II.1.1 Lemma
If $X, Y \subseteq M$ we have

$$S(\overline{X}) = S(X)$$

and

$$S(XY) = S(X)S(Y). \square$$

If X is a sub-monoid of M, then A(X) is a quotient coalgebra of A and S(X) a subalgebra of S. In this case A(X) is an (S(X),S(X))-bimodule, and we will write the associated actions as:

$$S(X) \times A(X) \to A(X)$$
$$(\xi, c) \mapsto \xi \circ c = (id \otimes \xi)(\nabla c)$$

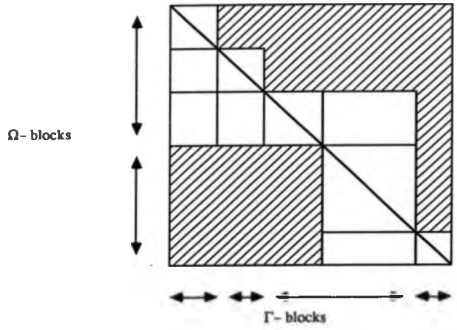
$$A(X) \times S(X) \to A(X)$$
$$(c, \xi) \mapsto c \circ \xi = (\xi \otimes id)(\nabla c).$$

II.1.2 The Definition of S(Ω,Γ)

Take subsets $\Omega, \Gamma \subseteq \Delta$, which will be fixed henceforth (unless stated otherwise), and define the following subset of $\underline{n} \times \underline{n}$:

$$[\Omega|\Gamma] = \{ (a,b) \in \underline{n} \times \underline{n} \,/ \begin{array}{l} a \sim_\Omega b \text{ if } a \geq b \\ a \sim_\Gamma b \text{ if } a \leq b \end{array} \}.$$

$[\Omega|\Gamma]$ can be represented as the non-shaded part of a diagram of the form:



For $\Omega,\Gamma \subseteq \Delta$ define a closed subgroup $P_{\Omega,\Gamma}$ of G by

$$P_{\Omega,\Gamma} = \{ g \in G \;/\; g_{a,b} = 0 \text{ unless } (a,b) \in [\Omega \,|\, \Gamma] \}.$$

Schematically $P_{\Omega,\Gamma}$ consists of all matrices in G which are zero in the shaded region of the diagram above. The $P_{\Omega,\Delta}$ are the parabolic subgroups of G containing the Borel subgroup $B^+ = P_{\emptyset,\Delta}$, whilst the $P_{\Delta,\Gamma}$ are the parabolic subgroups containing $B^- = P_{\Delta,\emptyset}$. We have $P_{\Omega,\Gamma} = P_{\Omega,\Delta} \cap P_{\Delta,\Gamma}$.

Write $A(\Omega,\Gamma)$ for $A(P_{\Omega,\Gamma})$ and $S(\Omega,\Gamma)$ for $S(P_{\Omega,\Gamma})$. The category mod $S(\Omega,\Gamma)$ is isomorphic to the category of finite-dimensional representations of the group $P_{\Omega,\Gamma}$ which are polynomial and homogeneous of degree f. (cf. [G1; §1, (2.2)].)

II.1.3 Lemma $\quad S(\Omega,\Gamma) = S(\Omega,\emptyset)S(\emptyset,\Gamma)$.

Proof

This will follow from II.1.1 once we know that $P_{\Omega,\emptyset}P_{\emptyset,\Gamma}$ is a Zariski dense subset of $P_{\Omega,\Gamma}$. In fact $P_{\Omega,\emptyset}P_{\emptyset,\Gamma}$ is the set of matrices in $P_{\Omega,\Gamma}$ whose leading minors are all non-zero. □

Remarks

(i) When $\Omega=\Gamma=\Delta$ the essential fact here is the density in GL_n of the 'Big Cell', see e.g. [St; Theorem 7] for a more general formulation of this result.

(ii) This lemma can also be proved directly within $S(\Omega,\Gamma)$ by a simple modification of [G2; §4] which treats the case $\Omega = \Gamma = \Delta$.

II.1.4 Bases for $A(\Omega,\Gamma)$ and $S(\Omega,\Gamma)$

$S(\Omega,\Gamma)$ has basis:

(a) $$\{\xi_{i,j} \;/\; i,j\in I(n,f),\ (i_\phi, j_\phi) \in [\Omega|\Gamma] \quad \forall \phi\in \underline{f}\}.$$

Hoping that no confusion will arise, we will denote the image of $c_{i,j}$ in $A(\Omega,\Gamma)$ by the same symbol. Then $A(\Omega,\Gamma)$ has basis:

(b) $$\{c_{i,j} \;/\; i,j\in I(n,f),\ (i_\phi, j_\phi) \in [\Omega|\Gamma] \quad \forall \phi\in \underline{f}\},$$

and this set consists of precisely those $c_{i,j}$ which are non-zero in $A(\Omega,\Gamma)$. These facts follow from the general remarks at the beginning of §1 since the ideal of functions vanishing on $P_{\Omega,\Gamma}$ is generated by the monomials $c_{a,b}$ with $(a,b) \notin [\Omega|\Gamma]$. The bases (a) and (b) are dual to one another, and the number of elements in each is $\binom{\#[\Omega|\Gamma]+f-1}{f}$.

II.1.5 Contravariant Duality

We will need a generalization of the notion of contravariant duality for the Schur algebra as set out in [G1; (2.7)]. In our case we will obtain a duality between the

categories mod S(Ω,Γ) and mod S(Γ,Ω). Let J be the algebra anti-automorphism of S induced by transposition in M, i.e. $J(\xi_{i,j}) = \xi_{j,i}$. If U is a subalgebra of S, the algebra anti-isomorphism $J: U \to J(U)$ identifies U^{op} with J(U), and hence gives rise to an isomorphism of module categories

$$J : \text{mod}'U \to \text{mod } J(U)$$
$$V \mapsto V^J \quad (V \in \text{mod}'U).$$

Composing this with the k-dual functor $\text{Hom}_k(\ ,k)$: mod U → mod'U, we get the ***contravariant duality*** functor:

$$\text{mod } U \to \text{mod } J(U)$$
$$V \mapsto V^0 \quad (V \in \text{mod } U).$$

This is an anti-equivalence of categories, and the composite

$$\text{mod } U \to \text{mod } J(U) \to \text{mod } J^2(U) = \text{mod } U$$

is isomorphic to the identity functor.

If $V \in \text{mod } U$ and $V' \in \text{mod } J(U)$, a ***contravariant form*** is a bilinear map

$$\langle\ ,\ \rangle : V \times V' \to k$$

satisfying

$$\langle uv, v' \rangle = \langle v, J(u)v' \rangle \quad \forall\, u \in U,\ v \in V,\ v' \in V'.$$

The natural isomorphism $\text{Hom}_k(V \otimes V', k) \cong \text{Hom}_k(V, \text{Hom}_k(V', k))$ takes the set of contravariant forms onto $\text{Hom}_U(V, (V')^0)$, and under this map, non-singular forms correspond to isomorphisms.

When U=S(Ω,Γ), we have J(U) = S(Γ,Ω), and there is a non-singular contravariant form

$$< \ , \ > : S(\Omega,\Gamma) \times A(\Gamma,\Omega) \to k$$
$$<\xi,c> = J(\xi)(c).$$

Thus $S(\Omega,\Gamma) \cong A(\Gamma,\Omega)^0$ as $S(\Omega,\Gamma)$-modules.

II.1.6 Weight Spaces

$S(\emptyset,\emptyset)$ is a basic semisimple k-algebra. For we have

$$S(\emptyset,\emptyset) = \bigoplus_{\lambda \in \Lambda(n,f)} k\cdot\xi_\lambda,$$

and $\xi_\mu\xi_\lambda = \delta_{\mu,\lambda}\cdot\xi_\mu \ \ \forall\, \mu,\lambda \in \Lambda(n,f)$, so $S(\emptyset,\emptyset) \cong k^N$ with $N = \#\Lambda(n,f)$.

For $\lambda \in \Lambda(n,f)$ define $\hat{\lambda} \in \mathrm{Hom}_k(S, k)$ to be evaluation on the element c_λ of A. Denote by $k(\lambda)$ the simple $S(\emptyset,\emptyset)$-module corresponding to the weight λ. Thus $k(\lambda)$ is the vector space k, with $S(\emptyset,\emptyset)$ acting by

$$\xi\cdot 1_k = \hat{\lambda}(\xi) \quad \forall\, \xi \in S(\emptyset,\emptyset).$$

$\{k(\lambda)\}_{\lambda \in \Lambda(n,f)}$ is a full set of simple $S(\emptyset,\emptyset)$-modules.

If $V \in \mathrm{mod}\, S(\emptyset,\emptyset)$ and $\lambda \in \Lambda(n,f)$ we write ${}^\lambda V$ for $\xi_\lambda\cdot V$, which is the sum of all the submodules of V which are isomorphic to $k(\lambda)$. We say that λ is a ***weight*** of V if ${}^\lambda V$ is non-zero. ${}^\lambda V$ is the ***λ-weight space*** of V [7], and its elements are called ***λ-weight vectors***. We have

$$V = \bigoplus_{\lambda \in \Lambda(n,f)} {}^\lambda V.$$

If $V \in \mathrm{mod}' S(\emptyset,\emptyset)$, V^λ is defined analogously, and similar remarks apply.

Any $V \in \mathrm{mod} S(\Omega,\Gamma)$ is an $S(\emptyset,\emptyset)$-module via the inclusion $S(\emptyset,\emptyset) \to S(\Omega,\Gamma)$,

[7] Any $S(\emptyset,\emptyset)$-module is a rational module for the torus T, and ${}^\lambda V$ as defined here is the λ-weight space in the usual sense for algebraic groups. (cf. [O1; (3.2)].)

so we can talk about its weights. Note that the weights of V and V^0 coincide.

II.1.7 Lemma $\quad \xi_{i,l} \in S(\Omega,\emptyset) \Rightarrow \mathrm{wt}(i) \trianglelefteq_\Omega \mathrm{wt}(l)$.

Proof

$\xi_{i,l} \in S(\Omega,\emptyset)$ iff $i \geq l$ and $i \sim_\Omega l$. Put $\mu = \mathrm{wt}(i)$, $\lambda = \mathrm{wt}(l)$. If $\Xi = \{a, a+1, \cdots\}$ is an Ω-block, and $b \in \Xi$, then $R_{b,c}(i,l) \neq \emptyset$ implies that $a \leq c \leq b$. Thus for each Ξ

$$\mu_a+\mu_{a+1}+\cdots+\mu_b \leq \lambda_a+\lambda_{a+1}+\cdots+\lambda_b \quad \forall b \in \Xi,$$

and

$$\sum_{b\in\Xi} \mu_b = \sum_{b\in\Xi} \lambda_b.$$

These conditions are easily checked to be equivalent to $\mu \trianglelefteq_\Omega \lambda$. □

II.1.8 Proposition

(i) ${}^{\mu}S(\Omega,\emptyset)^{\lambda}$ is zero unless $\mu \trianglelefteq_\Omega \lambda$.

(ii) $\dim {}^{\lambda}S(\Omega,\emptyset)^{\lambda} = 1$.

(iii) $\mathrm{rad}\, S(\Omega,\emptyset) = \bigoplus_{\mu<\lambda} {}^{\mu}S(\Omega,\emptyset)^{\lambda}$.

(iv) $S(\Omega,\emptyset) \,/\, \mathrm{rad}\, S(\Omega,\emptyset) \cong S(\emptyset,\emptyset)$ as algebras.

(v) $\{k(\lambda)\}_{\lambda\in\Lambda(n,r)}$ is a full set of simple $S(\Omega,\emptyset)$-modules, where the action of $S(\Omega,\emptyset)$ on $k(\lambda)$ is again given by $\bar{\lambda}$.

(vi) ${}^{\lambda}S(\Omega,\emptyset)^{\lambda}$ is the $S(\Omega,\emptyset)$-projective cover of $k(\lambda)$.

Proof

(i) follows from II.1.7. (ii) is clear. By (i) $\bigoplus_{\mu<\lambda} {}^{\mu}S(\Omega,\emptyset)^{\lambda}$ is a nilpotent ideal of $S(\Omega,\emptyset)$, whose quotient is isomorphic to the semisimple algebra $S(\emptyset,\emptyset)$ by (ii). This implies (iii), (iv) and (v). The decomposition

$$S(\Omega,\emptyset) = \bigoplus_{\lambda} S(\Omega,\emptyset)^{\lambda}$$

shows that ${}^{\lambda}S(\Omega,\emptyset)^{\lambda}$ is projective, and from (ii) and (iii) we infer that its head is $k(\lambda)$, giving (vi). □

II.1.9 W_Ψ as a subgroup of $S(\Psi,\Psi)$.

Take $\Psi\subseteq\Delta$. We obtain an image of W_Ψ in the algebra $S(\Psi,\Psi)$ by sending $w\in W_\Psi$ to $e(M(w))$, where $M(w)$ is the permutation matrix given by $M(w)_{a,b} = \delta_{a,wb}$ $(a,b\in\underline{n})$, and e is the evaluation map defined at the start of §1. The representation of W_Ψ on $S(\Psi,\Psi)$ by right multiplication obtained in this way is easily seen to be faithful [8], so in fact we have an embedding of W_Ψ in $S(\Psi,\Psi)$, which we will sometimes view as an identification.

Lemma

If $w\in W$, $i,j\in I(n,f)$, then

(i) $w\cdot\xi_{i,j} = \xi_{wi,j}$.

(ii) $\xi_{i,j}\cdot w = \xi_{i,w^{-1}j}$.

(iii) $w\cdot\xi_\lambda\cdot w^{-1} = \xi_{w\lambda}$.

Proof

$$(w\cdot\xi_{i,j})(c_{l,m}) = \sum_{r\in I(n,f)} c_{l,r}(M(w))\xi_{i,j}(c_{r,m})$$

$$= \xi_{i,j}(c_{w^{-1}l,m}) = \xi_{wi,j}(c_{l,m}).$$

This establishes (i). (ii) is similar, and (iii) follows from (i) and (ii). □

We see from the lemma that if $V\in\mathrm{mod}S(\Psi,\Psi)$, $w\in W_\Psi$, $\lambda\in\Lambda(n,f)$ then the action of w on V maps ${}^\lambda V$ onto ${}^{w\lambda}V$, so the set of weights of any $S(\Psi,\Psi)$-module is closed to the action of W_Ψ on $\Lambda(n,f)$. (cf. [G1; (3.3a)].)

§2: The Modules $V(\Omega,\Gamma)$ and $D(\Omega,\Gamma)$

In this section we define modules $V(\Omega,\Gamma,\lambda)$ and $D(\Omega,\Gamma,\lambda)$, which are analogues for the algebra $S(\Omega,\Gamma)$ of the Weyl and Schur modules for the classical Schur algebra. We work out some preliminary properties of $V(\Omega,\Gamma,\lambda)$ and $D(\Omega,\Gamma,\lambda)$ which will

[8] If $w\in W_\Psi$, $a\in\underline{n}$, then $w\,\xi_{i(a),j(a)} = \xi_{i(wa),j(a)}$, where for $b\in\underline{n}$, $i(b)$ denotes the index all of whose entries are equal to b.

enable us to classify the simple modules for S(Ω,Γ).

II.2.1 Definition

For the rest of this chapter fix $\lambda \in \Lambda(n,t)$, and define modules for S(Ω,Γ) by

$$V(\Omega,\Gamma,\lambda) = S(\Omega,\Gamma) \underset{S(\emptyset,\Gamma)}{\otimes} k(\lambda),$$

$$D(\Omega,\Gamma,\lambda) = \mathrm{Hom}_{S(\Omega,\emptyset)}(S(\Omega,\Gamma), k(\lambda)).$$

V(Ω,Γ,λ) and D(Ω,Γ,λ) are respectively the ***Weyl module*** and the ***Schur module*** associated with the weight λ. When Ω=Γ=Δ and λ is dominant this definition agrees with the usual one [9].

We will often use without further comment the natural isomorphism of k-spaces $A(\Omega,\Gamma) \cong \mathrm{Hom}_k(S(\Omega,\Gamma), k)$ to identify D(Ω,Γ,λ) with a submodule of A(Ω,Γ).

II.2.2 Lemma

Let K be the kernel of the map

$$S(\Omega,\Gamma) \to V(\Omega,\Gamma,\lambda)$$
$$\xi \mapsto \xi \otimes 1.$$

Then if $c \in A(\Gamma,\Omega)$ we have $\langle K,c\rangle = 0$ iff $c \in D(\Gamma,\Omega,\lambda)$. Thus $\langle\ ,\ \rangle$ induces a non-singular contravariant form $V(\Omega,\Gamma,\lambda) \times D(\Gamma,\Omega,\lambda) \to k$, which we will denote by the same symbol. In particular $V(\Omega,\Gamma,\lambda) \cong D(\Gamma,\Omega,\lambda)^o$.

Proof

Put $K^\perp = \{\ c \in A(\Gamma,\Omega)\ /\ \langle K,c\rangle = 0\ \}$. By the definition of $V(\Omega,\Gamma,\lambda) = S(\Omega,\Gamma) \underset{S(\emptyset,\Gamma)}{\otimes} k(\lambda)$ as a certain quotient of $S(\Omega,\Gamma) \underset{k}{\otimes} k(\lambda)$, K is S(Ω,Γ)-generated by the set

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9 II.2.6 and II.3.2 together show that $D(\Delta,\Delta,\lambda)$ is the module $D_{\lambda,K}$ of [G1; §4]. II.2.2 shows that $V(\Delta,\Delta,\lambda)$ is the contravariant dual of $D(\Delta,\Delta,\lambda)$, hence it is isomorphic to the module $V_{\lambda,K}$ of [G1; §5].

$$\{\zeta-\tilde{\lambda}(\zeta)1 \;/\; \zeta \in S(\emptyset,\Gamma)\}.$$

Therefore $c \in K^{\perp}$

$$\text{iff } \langle \xi(\zeta-\tilde{\lambda}(\zeta)1), c\rangle = 0 \;\; \forall\, \xi\in S(\Omega,\Gamma)\,,\, \zeta\in S(\emptyset,\Gamma)$$
$$\text{iff } c(J(\zeta)J(\xi)) = \tilde{\lambda}(\zeta)\, c(J(\xi)) \;\; \forall\, \xi\in S(\Omega,\Gamma)\,,\, \zeta\in S(\emptyset,\Gamma)$$
$$\text{iff } c(\zeta'\xi') = \tilde{\lambda}(\zeta')\cdot c(\xi') \;\; \forall\, \xi'\in S(\Gamma,\Omega)\,,\, \zeta'\in S(\Gamma,\emptyset)$$
$$\text{iff } c \in D(\Gamma,\Omega,\lambda). \;\square$$

Remark

The isomorphism $V(\Omega,\Gamma,\lambda) \cong D(\Gamma,\Omega,\lambda)^{o}$ can also be obtained by noting that there is an isomorphism of functors

$$\mathrm{Hom}_{S(\Gamma,\emptyset)}(S(\Gamma,\Omega),\;\;) \cong \big(\, S(\Omega,\Gamma) \underset{S(\emptyset,\Gamma)}{\otimes} ?^{o}\big)^{o}$$

This can be seen by using the fact that the functors $S(\Omega,\Gamma) \underset{S(\emptyset,\Gamma)}{\otimes}$ and $\mathrm{Hom}_{S(\Gamma,\emptyset)}(S(\Gamma,\Omega),\;\;)$ are respectively the left and right adjoints to the restriction functors $\mathrm{mod}S(\Omega,\Gamma) \to \mathrm{mod}S(\emptyset,\Gamma)$ and $\mathrm{mod}S(\Gamma,\Omega) \to \mathrm{mod}S(\Gamma,\emptyset)$.

II.2.3 Bideterminants

Henceforth Ψ will denote the intersection $\Omega\cap\Gamma$ [10]. We let 1 be the canonical index of weight λ, and use the notation of I.1.5 for our fixed weight λ. In II.2.6 we introduce a submodule $D'(\Gamma,\Omega,\lambda)$ of $A(\Gamma,\Omega)$ as a generalization of the module $D_{\lambda,K}$ of [G1; §4]. $D'(\Gamma,\Omega,\lambda)$ is defined as the k-span of certain elements which we might call Ψ-bideterminants. Our aim in introducing $D'(\Gamma,\Omega,\lambda)$ is to provide a lower bound on the dimensions of $V(\Omega,\Gamma,\lambda)$ and $D(\Gamma,\Omega,\lambda)$ when λ is Ψ-dominant. This estimate will later be refined to an equality, and it will follow that in fact $D(\Gamma,\Omega,\lambda)$ and $D'(\Gamma,\Omega,\lambda)$ are identical for Ψ-dominant λ. This is not the case for non-dominant λ — see the remark following II.2.7.

Bearing in mind our convention about $c_{i,j} \in A(\Gamma,\Omega)$ (see II.1.4), define

---

[10] $P_{\Psi,\Psi}$ is the largest standard Levi subgroup contained in $P_{\Omega,\Gamma}$.

$$d^{\lambda}_{i,j} = d_{i,j} = \sum_{\pi \in C(\lambda,\Psi)} \operatorname{sgn}(\pi) \cdot c_{i,j\pi}$$

$$= \sum_{\pi \in C(\lambda,\Psi)} \operatorname{sgn}(\pi) \cdot c_{i\pi,j} \in A(\Gamma,\Omega).$$

It is perhaps worth emphasizing that $d_{i,j}$ depends on Ω,Γ and λ.

The following results are easily verified:

II.2.4 Lemma

(i) $\nabla d_{i,j} = \sum_{m \in I(n,f)} d_{i,m} \otimes c_{m,j} = \sum_{m \in I(n,f)} c_{i,m} \otimes d_{m,j}$.

(ii) If $\pi \in C(\lambda,\Psi)$ then $d_{i\pi,j} = d_{i,j\pi} = \operatorname{sgn}(\pi) \cdot d_{i,j}$.

(iii) If either i or j has repeated entries in some Ψ-column then $d_{i,j} = 0$. □

We will also need:

II.2.5 Lemma

Suppose $i,m \in I(n,f)$ with $l \sim_\Omega i$. Then

(i) $l\pi \sim_\Omega i \quad \forall\, \pi \in C(\lambda,\Psi)$.

(ii) If $d_{l,m}$, $c_{m,i}$ are both non-zero, then $l \sim_\Omega m$.

(iii) If $c_{l,m}$, $d_{m,i}$ are both non-zero, then $l \sim_\Omega m$.

Proof

Take $\pi \in C(\lambda,\Psi)$ and $\varphi \in f$. By definition of l we have $l\pi_\varphi \sim_\Psi l_\varphi \sim_\Omega i_\varphi$, and so (i) follows. Now suppose that the hypotheses of (ii) hold, and that $\pi \in C(\lambda,\Psi)$ has been chosen so that $c_{l\pi,m}$ is non-zero. Consider cases:

If $l\pi_\varphi \le m_\varphi$ then $l_\varphi \sim_\Psi l\pi_\varphi \sim_\Omega m_\varphi$, for $c_{l\pi,m} \neq 0$ implies $(l\pi_\varphi, m_\varphi) \in [\Gamma\backslash\Omega]$.

If $m_\varphi \le i_\varphi$ then $m_\varphi \sim_\Omega i_\varphi \sim_\Omega l_\varphi$.

If $l\pi_\varphi \ge m_\varphi \ge i_\varphi$ then $l\pi_\varphi \sim_\Omega i_\varphi$ implies $l_\varphi \sim_\Psi l\pi_\varphi \sim_\Omega m_\varphi$.

The proof of (iii) is similar. □

II.2.6 Definition

Define $D'(\Gamma,\Omega,\lambda)$ to be the k-span of the set $\{d_{l,i} \ / \ l \sim_\Omega i\}$. We see from II.2.4(i) and II.2.5(ii) that $D'(\Gamma,\Omega,\lambda)$ is an $S(\Gamma,\Omega)$-submodule of $A(\Gamma,\Omega)^\lambda$.

II.2.7 Proposition

If λ is Ψ-dominant then $D'(\Gamma,\Omega,\lambda) \leq D(\Gamma,\Omega,\lambda)$.

Proof

Let $c \in A(\Gamma,\Omega)$. If $\zeta \in S(\Gamma,\varnothing)$, $\xi \in S(\Gamma,\Omega)$ then $c(\zeta\xi) = (\zeta \otimes \xi)(\nabla c) = (c \circ \zeta)(\xi)$, so

$$c \in D(\Gamma,\Omega,\lambda) \text{ iff } c \circ \zeta = \bar{\lambda}(\zeta) \cdot c \ \ \forall \ \zeta \in S(\Gamma,\varnothing).$$

$S(\Gamma,\varnothing)$ has basis $\{ \xi_{r,s} \ / \ r \geq s \text{ and } r \sim_\Gamma s \}$, so it suffices to check that if $l \sim_\Omega l$ then

(i) $$d_{l,i} \circ \xi_\lambda = d_{l,i},$$

(ii) $$d_{l,i} \circ \xi_{r,s} = 0 \quad \text{if } r \geq s,\ r \sim_\Gamma s, \text{ and } (r,s) \neq (l,l).$$

Since $d_{l,i} \in A(\Gamma,\Omega)^\lambda$, (i) is clear. By II.2.4(i) we have

$$d_{l,i} \circ \xi_{r,s} = \sum_m \xi_{r,s}(c_{l,m}) \cdot d_{m,i},$$

which is zero unless $r \sim l$. Thus suppose $r = l > s$, and, for a contradiction, that there is some $m \in I(n,f)$ with both $\xi_{l,s}(c_{l,m})$ and $d_{m,i}$ non-zero. Then $l > m$ and $l \sim_\Gamma m$. By II.2.5(iii) we also have $l \sim_\Omega m$, and since the Ψ-blocks are simply the various intersections of the Ω-blocks with the Γ-blocks, we have $l \sim_\Psi m$. Choose $a \in \mathbb{N}$ to be minimal such that

$$m_\varphi < l_\varphi = a \text{ for some } \varphi \in R_a.$$

Let C_b be the column containing φ, and put $\Xi = W_\Psi \cdot a$. Then $m_\varphi \in \Xi$ since $l \sim_\Psi m$. The Ψ-dominance of λ implies that $R_{m_\varphi} \cap C_{\Xi,b}$ is non-empty, say $R_{m_\varphi} \cap C_{\Xi,b} = \{\varphi'\}$. Now $\varphi, \varphi' \in C_{\Xi,b}$, $\varphi \neq \varphi'$, and, by minimality of a, $m_{\varphi'} = l_{\varphi'} = m_\varphi$. Thus m has

repeated entries in $C_{\Xi,b}$, so by II.2.4(iii), $d_{m,i} = 0$, contrary to assumption. □

Remark

This proposition is false for weights λ which are not Ψ-dominant. We will see in II.2.12 that D(Γ,Ω,λ) is non-zero iff λ is Ψ-dominant. However D'(Γ,Ω,λ) is always non-zero; in fact D'(Γ,Ω,λ) ≅ D'(Γ,Ω,μ) whenever λ and μ lie in the same W_Ψ-orbit. To see this take $w \in W_\Psi$ and let $\omega \in P(f)$ be the composite

$$\underset{(\mathbb{T}^\lambda)^{-1}}{\underset{}{f} \to} [\lambda] \to [w\lambda] \underset{\mathbb{T}^{w\lambda}}{\to} f,$$

where the middle map is given by $(a,b) \mapsto (wa,b)$. Then ω maps the λ-Ψ-columns to the wλ-Ψ-columns, and hence $C(w\lambda,\Psi) = \omega C(\lambda,\Psi)\omega^{-1}$. The map $c_{I,i} \mapsto c_{wI,i}$ is an $S(\Gamma,\Omega)$-isomorphism $A(\Gamma,\Omega)^\lambda \to A(\Gamma,\Omega)^{w\lambda}$. In fact it maps D'(Γ,Ω,λ) onto D'(Γ,Ω,wλ), for $wI\omega^{-1}$ is canonical of weight wλ, $I \sim_\Omega i$ iff $wI\omega^{-1} \sim_\Omega i\omega^{-1}$, and

$$d^{w\lambda}_{wI\omega^{-1},i\omega^{-1}} = \sum_{\pi \in C(w\lambda,\Psi)} \lfloor c_{wI\omega^{-1}\pi,i\omega^{-1}} = \sum_{\pi \in C(\lambda,\Psi)} \lfloor c_{wI\omega^{-1}\omega\pi\omega^{-1},i\omega^{-1}}$$

$$= \sum_{\pi \in C(\lambda,\Psi)} \lfloor c_{wI\pi,i}$$

which is the image of $d^\lambda_{I,i}$.

II.2.8 Standardness of $\xi_{i,I}$

We will say that a basis element $\xi_{i,I}$ is *Ψ-standard* if $\xi_{i,I} = \xi_{j,I}$ for some j which is Ψ-column standard. Suppose that λ is Ψ-dominant and let j be the index obtained from i by reordering the rows so that they become semi-standard. Then by I.1.6, $\xi_{i,I}$ is Ψ-standard iff j is Ψ-column standard.

II.2.9 Lemma

Suppose that $\xi \in S(\Omega,\varnothing)^{\lambda}$ is a linear combination of Ψ-standard $\xi_{i,l}$. Then

$$\langle \xi, D'(\Gamma,\Omega,\lambda)\rangle = 0 \Rightarrow \xi = 0.$$

Proof

Firstly we show that if $j \in I(n,\Gamma)$ is Ψ-column standard, then

(a) $\quad d_{l\,j} = c_{l,j} +$ a linear combination of terms $c_{l,m}$ with $m >_l j$.

Suppose $1 \neq \pi \in C(\lambda,\Psi)$, and put $m = j\pi$. Choose $a \in \underline{n}$ to be minimal with $\pi(R_a) \neq R_a$, and let $\Xi = W_{\Psi} \cdot a$. Let $\alpha = wt(\, j \mid_{R_a})$, $\beta = wt(\, m \mid_{R_a})$. We must show that $\alpha >_{lex} \beta$. Suppose that $\rho \in R_a \cap C_b$ $(b \in \underline{n})$ and $x\rho \notin R_a$. Since $\rho < x\rho \in C_{\Xi,b}$, and j is Ψ-column standard, $j_\rho < j\pi_\rho = m_\rho$. Thus if we choose ρ so that $x\rho \notin R_a$, and j_ρ is as small as possible, we have

$$\alpha_c = \beta_c \qquad \forall c < j_\rho,$$

and

$$\alpha_{j_\rho} > \beta_{j_\rho},$$

as required.

Now suppose that $0 \neq \xi \in S(\Omega,\varnothing)^{\lambda}$ is a linear combination of Ψ-standard $\xi_{i,l}$, and write

$$\xi = \sum_{a} \alpha(a)\cdot\xi_{i(a),l} \qquad (\,0 \neq \alpha(a) \in k\,),$$

with the $i(a)$ Ψ-column standard, and ordered so that $a < b \Rightarrow i(a) >_l i(b)$. Since $\xi \in S(\Omega,\varnothing)$ we have $i(a) \sim_{\Omega} l$, so $d_{l\,i(a)} \in D'(\Gamma,\Omega,\lambda)$. Consider

$$x(a) = \langle \xi_{i(a),l}, d_{l,i(1)}\rangle = \sum_{\pi \in C(\lambda,\Psi)} \underbrace{\xi_{l,i(a)}}_{\text{sgn}^*(\pi)} (c_{l,i(1)}\pi).$$

If $c_{1,j}$ appears with non-zero coefficient in $d_{1,i(1)}$ we have

$$j \geq_1 i(1) \geq_1 i(a)$$

by (a). If $a > 1$ we have strict inequality, so $x(a) = 0$. If $a=1$, we again get strict inequality unless $j=i(1)$. By (a) $c_{1,i(1)}$ appears with coefficient 1 in $d_{1,i(1)}$, so $x(1) = 1$, and

$$\langle \xi, d_{1,i(1)} \rangle = \sum_a \alpha(a)x(a) = \alpha(1) \neq 0.$$

Thus $\langle \xi , D'(\Gamma,\Omega,\lambda) \rangle \neq 0$. □

II.2.10 Proposition

Suppose that λ is Ψ-dominant. Then the images of the Ψ-standard $\xi_{i,1} \in S(\Omega,\varnothing)^\lambda$ under the map

$$h: S(\Omega,\Gamma) \to V(\Omega,\Gamma,\lambda)$$
$$\xi \mapsto \xi \otimes 1$$

are linearly independent. In particular, since $\xi_{1,1}$ is Ψ-standard $V(\Omega,\Gamma,\lambda) \neq 0$.

Proof

If $\xi \in S(\Omega,\varnothing)^\lambda$ is a linear combination of Ψ-standard $\xi_{i,1}$, with $h(\xi) = 0$, we have $\langle \xi, D'(\Gamma,\Omega,\lambda) \rangle = 0$ by II.2.2 and II.2.7. Thus $\xi = 0$ by II.2.9. □

We can now classify the simple $S(\Omega,\Gamma)$-modules.

II.2.11 Theorem

(i) $V(\Omega,\Gamma,\lambda) \neq 0$ iff λ is Ψ-dominant.

Suppose that λ is Ψ-dominant. Then

(ii) $S(\Omega,\varnothing)^\lambda$ is the $S(\Omega,\varnothing)$-projective cover of $V(\Omega,\Gamma,\lambda)$.

(iii) $V(\Omega,\Gamma,\lambda)$ has simple $S(\Omega,\Gamma)$-head, which we will denote $L(\Omega,\Gamma,\lambda)$.

(iv) If μ is a weight of $V(\Omega,\Gamma,\lambda)$ or $L(\Omega,\Gamma,\lambda)$ then $\mu \leq_\Omega \lambda$, and

$$\dim {}^\lambda V(\Omega,\Gamma,\lambda) = \dim {}^\lambda L(\Omega,\Gamma,\lambda) = 1.$$

(v) The $L(\Omega,\Gamma,\lambda)$ for λ ranging over all Ψ-dominant weights form a complete set of pairwise non-isomorphic simple $S(\Omega,\Gamma)$-modules. Each $L(\Omega,\Gamma,\lambda)$ is absolutely irreducible.

Proof

If λ is Ψ-dominant, $V(\Omega,\Gamma,\lambda) \neq 0$ by II.2.10. Suppose that $V(\Omega,\Gamma,\lambda) \neq 0$. Since $S(\Omega,\Gamma) = S(\Omega,\varnothing)S(\varnothing,\Gamma)$ by II.1.3, $V(\Omega,\Gamma,\lambda)$ is $S(\Omega,\varnothing)$-generated by the λ-weight vector $1\otimes 1$, and hence the composite map

$$S(\Omega,\varnothing)^\lambda \to S(\Omega,\Gamma) \to V(\Omega,\Gamma,\lambda)$$

is an epimorphism. $S(\Omega,\varnothing)^\lambda$ is an indecomposable projective in mod $S(\Omega,\varnothing)$ by II.1.8, so (ii) holds. In particular $V(\Omega,\Gamma,\lambda)$ has simple $S(\Omega,\varnothing)$-head, so its $S(\Omega,\Gamma)$-head is certainly simple, giving (iii). Statement (iv) about weights follows from II.1.8. The group W_Ψ acts on the weights of any $S(\Omega,\Gamma)$-module as in II.1.9, so we see that

$$w\lambda \trianglelefteq \lambda \quad \forall\, w \in W_\Psi,$$

and hence λ is Ψ-dominant, completing the proof of (i). The $L(\Omega,\Gamma,\lambda)$ are certainly pairwise non-isomorphic since by (iv) they have different weight structures. By (ii) $L(\Omega,\Gamma,\lambda)$ is generated by the one-dimensional weight space ${}^\lambda L(\Omega,\Gamma,\lambda)$. Any endomorphism of $L(\Omega,\Gamma,\lambda)$ maps this weight space into itself so $\mathrm{End}_{S(\Omega,\Gamma)} L(\Omega,\Gamma,\lambda) \cong k$, i.e. $L(\Omega,\Gamma,\lambda)$ is absolutely irreducible. Finally, suppose that $V \in \mathrm{mod}\, S(\Omega,\Gamma)$ is simple. Then

$$\mathrm{Hom}_{S(\varnothing,\Gamma)}(k(\lambda), V) \neq 0 \text{ for some weight } \lambda.$$

By adjointness

$$\mathrm{Hom}_{S(\Omega,\Gamma)}(V(\Omega,\Gamma,\lambda), V) \neq 0,$$

so $V \cong L(\Omega,\Gamma,\lambda)$ by simplicity of V, proving (v). □

Remarks

(i) The Weyl modules $V(\Omega,\Delta,\lambda)$ for the parabolic Schur algebra $S(\Omega,\Delta)$ are considered in [Sa]. There II.2.11(i) is proved for $V(\Omega,\Delta,\lambda)$ by analysing the restriction of $V(\Omega,\Delta,\lambda)$ to the algebra $S(\Omega,\Omega)$, and thereby reducing the problem to the corresponding theorem for the classical situation. The above approach to the simple modules appears in [Sa] for the case $S(G) = S(\Delta,\Delta)$.

(ii) The proof of the last part of the theorem shows that the $S(\emptyset,\Gamma)$-socle of any $S(\Omega,\Gamma)$-module contains only Ψ-dominant weights.

(iii) Since contravariant duality preserves weights

$$L(\Omega,\Gamma,\lambda)^{\circ} \cong L(\Gamma,\Omega,\lambda),$$

because both are simple modules with highest weight λ. We can thus strengthen part (iv) of the theorem: if μ is a weight of $L(\Omega,\Gamma,\lambda)$ then $\mu \leq_{\Psi} \lambda$.

By applying contravariant duality to II.2.11 we can read off:

II.2.12 Theorem

(i) $D(\Gamma,\Omega,\lambda) \neq 0$ iff λ is Ψ-dominant.

Suppose that λ is Ψ-dominant. Then

(ii) $A(\emptyset,\Omega)^{\lambda}$ is the $S(\emptyset,\Omega)$-injective hull of $D(\Gamma,\Omega,\lambda)$.

(iii) $D(\Gamma,\Omega,\lambda)$ has simple socle $L(\Gamma,\Omega,\lambda)$.

(iv) If μ is a weight of $D(\Gamma,\Omega,\lambda)$ then $\mu \leq_{\Omega} \lambda$, and $\dim {}^{\lambda}D(\Gamma,\Omega,\lambda) = 1$. □

II.2.13 Characterizations of Weyl Modules [11]

Let λ be a Ψ-dominant weight. We close this section with a couple of characterizations of the module $V(\Omega,\Gamma,\lambda)$. By (the transposed version of) II.1.8 the simple $S(\emptyset,\Gamma)$-module $k(\lambda)$ has a projective presentation of the form

[11] Similar characterizations apply to the Schur modules.

$$\coprod_{c} S(\emptyset,\Gamma)^{\mu(c)} \to S(\emptyset,\Gamma)^{\lambda} \to k(\lambda) \to 0,$$

where the weights $\mu(c)$ all satisfy $\lambda \lhd \mu(c)$ [12]. Applying the right exact functor $S(\Omega,\Gamma) \underset{S(\emptyset,\Gamma)}{\otimes}$ we get a presentation of $V(\Omega,\Gamma,\lambda)$:

(a) $$\coprod_{c} S(\Omega,\Gamma)^{\mu(c)} \to S(\Omega,\Gamma)^{\lambda} \to V(\Omega,\Gamma,\lambda) \to 0.$$

If $w \in W_\Psi$, multiplication on the right by w^{-1} gives an isomorphism $S(\Omega,\Gamma)^{\mu} \cong S(\Omega,\Gamma)^{w\mu}$, so we may suppose that the weights $\mu(c)$ appearing in (a) are all Ψ-dominant. This does not change the fact that $\lambda \lhd \mu(c)$ for all c.

Using (a) and induction on the dominance order, we see that if $L(\Omega,\Gamma,\mu)$ appears in the head of $S(\Omega,\Gamma)^{\lambda}$, then $\lambda \trianglelefteq \mu$; $L(\Omega,\Gamma,\lambda)$ itself appears with multiplicity one. Therefore by (a) ***$V(\Omega,\Gamma,\lambda)$ is the unique largest quotient of $S(\Omega,\Gamma)^{\lambda}$ all of whose weights are dominated by λ***.

If $U \leq U'$ are k-algebras we will say that $V \in \mathrm{mod}\ U$ ***extends to a U'-module*** if there is a U'-module structure on V whose restriction to U is the given U-module structure. Suppose $\Gamma' \subseteq \Gamma$. The obvious map

$$V(\Omega,\Gamma',\lambda) = S(\Omega,\Gamma') \underset{S(\emptyset,\Gamma')}{\otimes} k(\lambda) \to S(\Omega,\Gamma) \underset{S(\emptyset,\Gamma)}{\otimes} k(\lambda) = V(\Omega,\Gamma,\lambda)$$

is an epi by II.1.3. Using the characterization of $V(\Omega,\Gamma,\lambda)$ just given, we see that ***$V(\Omega,\Gamma,\lambda)$ is the unique largest quotient of $V(\Omega,\Gamma',\lambda)$ which extends to an $S(\Omega,\Gamma)$-module***. In fact there is a unique extension of the $S(\Omega,\Gamma')$-module $V(\Omega,\Gamma,\lambda)$ to an $S(\Omega,\Gamma)$-module, since the restriction of $V(\Omega,\Gamma,\lambda)$ to $S(\Omega,\Gamma')$ determines its weight structure and the fact that it is generated by a λ-weight vector. Taking $\Gamma' = \emptyset$ shows that $V(\Omega,\Gamma,\lambda)$ is the largest quotient of $S(\Omega,\emptyset)^{\lambda}$ which extends to an $S(\Omega,\Gamma)$-module.

[12] The weights $\mu(c)$ are not necessarily distinct for distinct c.

§3: S(Ω,Γ) as a Quasi-Hereditary Algebra

In II.2.10 we showed that if λ is Ψ-dominant the images of the Ψ-standard $\xi_{i,j} \in S(\Omega,\varnothing)^{\lambda}$ are linearly independent in $V(\Omega,\Gamma,\lambda)$. In the next chapter (see III.3.1) we will prove the following result:

II.3.1 Theorem

Suppose that λ is Ψ-dominant. Then the images of the Ψ-standard $\xi_{i,j} \in S(\Omega,\varnothing)^{\lambda}$ under the map

$$S(\Omega,\Gamma) \to V(\Omega,\Gamma,\lambda)$$
$$\xi \mapsto \xi \otimes 1$$

form a basis of $V(\Omega,\Gamma,\lambda)$. In particular the dimension of $V(\Omega,\Gamma,\lambda)$ is independent of the ground field k.

We deduce immediately the following:

II.3.2 Corollary

Suppose λ is Ψ-dominant, and let l be the canonical index of weight λ. Then $D'(\Gamma,\Omega,\lambda) = D(\Gamma,\Omega,\lambda)$, and this module has a basis consisting of the elements $d_{l,i}$ for row semi-standard, Ψ-column standard indices i with $i \geq l$ and $i \sim_{\Omega} l$.

Proof

Let X be the subspace of $D'(\Gamma,\Omega,\lambda) \leq D(\Gamma,\Omega,\lambda)$ spanned by the $d_{l,i}$ satisfying the above conditions. III.3.1 together with the proof of II.2.9 shows that the map $V(\Omega,\Gamma,\lambda) \to \mathrm{Hom}_{\mathbf{k}}(X, \mathbf{k})$ induced by the contravariant form

$$< , >: V(\Omega,\Gamma,\lambda) \times D(\Gamma,\Omega,\lambda) \to \mathbf{k}$$

is injective. Hence

$$\dim V(\Omega,\Gamma,\lambda) \leq \dim X \leq \dim D'(\Gamma,\Omega,\lambda) \leq \dim D(\Gamma,\Omega,\lambda).$$

Since $V(\Omega,\Gamma,\lambda)$ and $D(\Gamma,\Omega,\lambda)$ are contravariant duals by II.2.2 we have equality

throughout in the above expression, and the result follows. □

We will now assume theorem II.3.1 to be proved and explore some of the consequences for the algebra $S(\Omega,\Gamma)$. We reassure the reader that nothing proved in this section will be used in the proof of II.3.1! Our main aim is to show that $S(\Omega,\Gamma)$ is a quasi-hereditary algebra in the sense of [Sc].

II.3.3 Quasi-Hereditary Algebras

We recall some definitions (see [DR]). Let U be a finite-dimensional k-algebra. An ideal J of U is called a ***heredity ideal*** if

(i) J is projective as a U-module;
(ii) $J^2 = J$;
(iii) $J \cdot \mathrm{rad}U \cdot J = 0$.

A ***heredity chain*** is a sequence

$$0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_t = U$$

of ideals such that for each $a \in t$ J_a / J_{a-1} is a heredity ideal in U / J_{a-1}. An algebra U is called a ***quasi-hereditary algebra*** if it possesses a heredity chain.

We refer to [DR] for the following: any idempotent ideal J in a finite-dimensional algebra U is idempotently generated, i.e. there is some idempotent $e \in U$ with $J = UeU$; if (iii) above holds for such an ideal then (i) is equivalent to the condition

(i') the multiplication map $Ue \underset{eUe}{\otimes} eU \to UeU = J$ is bijective.

We now adapt the proof in [P; §4] that the classical Schur algebra is quasi-hereditary. Order the Ψ-dominant weights in $\Lambda(n,f)$ as $\lambda(1), \lambda(2), \cdots, \lambda(t)$, in such a

way that $\lambda(a) \trianglelefteq \lambda(b)$ implies $a \geq b$. Let e_a be the idempotent $\xi_{\lambda(1)} + \cdots + \xi_{\lambda(a)}$ of $S(\emptyset,\emptyset)$, and let J_a be the ideal of $S(\Omega,\Gamma)$ generated by e_a. We will show that:

II.3.4 Theorem

$$0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_t = S(\Omega,\Gamma)$$

is a heredity chain in $S(\Omega,\Gamma)$, so $S(\Omega,\Gamma)$ is a quasi hereditary algebra. Thus $S(\Omega,\Gamma)$ has finite global dimension (see e.g. [DR]). For each $a \in t$ the section J_a / J_{a-1} is isomorphic as an $(S(\Omega,\Gamma),S(\Omega,\Gamma))$-bimodule to $V(\Omega,\Gamma,\lambda(a)) \otimes V(\Gamma,\Omega,\lambda(a))^J$ [13], where J is the transposition functor of II.1.5.

II.3.5 Organization of the proof of II.3.4

The proof of II.3.4 will be accomplished in several stages, which we outline here. In II.3.6 we follow [P] and show that there is an epimorphism

$$V(\Omega,\Gamma,\lambda(a)) \otimes V(\Gamma,\Omega,\lambda(a))^J \twoheadrightarrow J_a / J_{a-1},$$

thereby reducing the problem to a dimension formula:

(a) $$\dim S(\Omega,\Gamma) = \sum_{\substack{\text{all } \Psi\text{-dominant} \\ \text{weights } \lambda}} \dim V(\Omega,\Gamma,\lambda) \dim V(\Gamma,\Omega,\lambda).$$

By II.3.1 we may suppose that the characteristic of k is zero. In II.3.7 we show that in characteristic zero the algebra $S(\Psi,\Psi)$ is semisimple, and use this fact in II.3.8 to establish (a) in the special case when $\Omega \supseteq \Gamma$. In II.3.9–II.3.13 we use a combinatoric argument to extend (a) to the general case.

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[13] In fact $V(\Gamma,\Omega,\lambda)^J \cong k(\lambda) \underset{S(\Omega,\emptyset)}{\otimes} S(\Omega,\Gamma)$, so regarding $k(\lambda)$ as an $(S(\emptyset,\Gamma),S(\Omega,\emptyset))$-bimodule in the obvious way we have

$$J_a / J_{a-1} \cong S(\Omega,\Gamma) \underset{S(\emptyset,\Gamma)}{\otimes} k(\lambda) \underset{S(\Omega,\emptyset)}{\otimes} S(\Omega,\Gamma).$$

II.3.6 Reduction to the Dimension Formula

Put $S_a = S(\Omega,\Gamma) / J_a$. To prove II.3.4 we show that for each $a \in t$, conditions (i'),(ii) and (iii) of II.3.3 hold for the ideal J_a / J_{a-1} of S_{a-1}. (ii) is immediate from the definition of J_a. As in II.2.13 $V(\Omega,\Gamma,\lambda(a))$ has a presentation

(a) $$\coprod_{c} S(\Omega,\Gamma)^{\mu(c)} \to S(\Omega,\Gamma)^{\lambda(a)} \to V(\Omega,\Gamma,\lambda(a)) \to 0.$$

where the weights $\mu(c)$ are Ψ-dominant and strictly dominate $\lambda(a)$.

A module $V \in \mathrm{mod} S(\Omega,\Gamma)$ lies in $\mathrm{mod}\, S_{a-1}$ iff its weight spaces for the weights $\lambda(1), \lambda(2), \cdots, \lambda(a-1)$ are all zero. Since all the weights of the simple module $L(\Omega,\Gamma,\mu)$ are dominated by μ, V lies in $\mathrm{mod}\, S_{a-1}$ iff its composition factors are all of the form $L(\Omega,\Gamma,\lambda(b))$ for $b \geq a$. In particular $V(\Omega,\Gamma,\lambda(a)) \in \mathrm{mod}\, S_{a-1}$. The functor $S_{a-1} \underset{S(\Omega,\Gamma)}{\otimes}$ takes any $S(\Omega,\Gamma)$-module to its largest quotient lying in $\mathrm{mod}\, S_{a-1}$. We have

$$S_{a-1} \underset{S(\Omega,\Gamma)}{\otimes} S(\Omega,\Gamma)^{\mu} \cong (S_{a-1})^{\mu},$$

which is zero whenever $\mu = \lambda(b)$ with $b < a$, so applying $S_{a-1} \underset{S(\Omega,\Gamma)}{\otimes}$ to the sequence (a) gives an isomorphism

(b) $$V(\Omega,\Gamma,\lambda(a)) \cong S_{a-1}\cdot\bar{e}_a.$$

where $\bar{e}_a$ denotes the image of e_a in S_{a-1} [14].

Interchanging Ω and Γ, and applying the transposition functor J gives an isomorphism

(c) $$V(\Gamma,\Omega,\lambda(a))^{J} \cong e_a\cdot S_{a-1}.$$

We can now prove that condition (iii) of II.3.3 holds. We must show that

[14] Since $V(\Omega,\Gamma,\lambda(a))$ is indecomposable, $\bar{e}_a$ is a primitive idempotent and $V(\Omega,\Gamma,\lambda(a))$ is the projective cover of $L(\Omega,\Gamma,\lambda(a))$ in the category $\mathrm{mod}\, S_{a-1}$.

$\bar{e}_a \cdot \mathrm{rad}\, S_{a-1} \cdot \bar{e}_a = 0$, or equivalently that $\bar{e}_a \cdot S_{a-1} \cdot \bar{e}_a \cong \mathrm{End}_{S_{a-1}}(\bar{e}_a \cdot S_{a-1})$ is a semi-simple k-algebra. This is certainly true since by II.2.11 (using the same argument which shows that $L(\Omega,\Gamma,\lambda)$ is absolutely irreducible) it is isomorphic to k.

To complete the proof of II.3.4 we must establish (i'), i.e. that the epimorphism

(d) $$S_{a-1} \cdot \bar{e}_a \underset{k}{\otimes} \bar{e}_a \cdot S_{a-1} \;\to\; S_{a-1} \cdot \bar{e}_a \cdot S_{a-1}$$

given by multiplication is an isomorphism. Using (b), (c) and (d) we have

(e) $$\dim S(\Omega,\Gamma) \;\le \sum_{\substack{\text{all } \Psi\text{-dominant} \\ \text{weights } \lambda}} \dim V(\Omega,\Gamma,\lambda) \cdot \dim V(\Gamma,\Omega,\lambda),$$

and (d) is an isomorphism for all $a \in \underline{t}$ iff (e) is an equality. This reduces the problem to proving the identity II.3.5(a).

II.3.7 Theorem

If $\mathrm{char}(k) = 0$, or $\mathrm{char}(k) > f$ then $S(\Psi,\Psi)$ is a semisimple k-algebra.

Proof

The following proof is a simple adaptation of the proof for the classical case given in [G3; Theorem VII]. Let E be an n-dimensional k-space with basis $e_1, e_2, \cdots, e_n$. E becomes a G-module in the usual way:

$$g \cdot e_b = \sum_{a=1}^{n} g_{a,b} \cdot e_a \qquad (g \in G,\ b \in \underline{n}).$$

$E^{\otimes f}$ is a homogeneous polynomial representation of G of degree f when G acts 'diagonally' and hence an $S = S_f$-module. $E^{\otimes f}$ has a basis $\{e_i\}_{i \in I(n,f)}$, where $e_i = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_f}$, and we will denote by $E_{i,j}$ the element of $\mathrm{End}_k(E^{\otimes f})$ whose matrix with respect to this basis has a 1 in the (i,j) position and zeroes elsewhere. If ω is a P(f)-orbit of $I(n,f) \times I(n,f)$, put $\theta_\omega = \sum_{(i,j) \in \omega} E_{i,j}$.

Let $\psi: S \to \mathrm{End}_k(E^{\otimes f})$ be the representation afforded by the S-module $E^{\otimes f}$. It is shown in [G1; (2.6c)] that ψ is a faithful representation of S and that if

$\omega = (i,j)\text{-}P(f)$, the image of $\xi_{i,j}$ under ψ is θ_ω. The image U of $S(\Psi,\Psi)$ under ψ has a basis consisting of the θ_ω for orbits $\omega = (i,j)\text{-}P(f)$ with $i \neq_\psi j$. Define a bilinear form on U by

$$(\theta,\varphi) = \text{trace}(\theta\varphi).$$

This form is non-degenerate: take a non-zero element $\theta = \sum_\omega \alpha(\omega)\theta_\omega \in U$. Suppose $\alpha(\nu) \neq 0$, and let ν' be the transposed orbit to ν, i.e. if $\nu = (i,j)\text{-}P(f)$, then $\nu' = (j,i)\text{-}P(f)$. Note that $\theta_{\nu'}$ is an element of U.

$$(\theta,\theta_{\nu'}) = \sum_\omega \alpha(\omega) \sum_{\substack{(i,j)\in\omega \\ (q,r)\in\nu'}} \text{trace}(E_{i,j}E_{q,r}) = \sum_\omega \alpha(\omega) \sum_{\substack{(i,j)\in\omega \\ (q,r)\in\nu'}} \delta_{i,r}\delta_{j,q} = \alpha(\nu)\text{-}|\nu|,$$

which is non-zero because $|\nu|$ is a divisor of the order of P(f), and under the given hypotheses this is non-zero in k.

Suppose that $\theta \in \text{rad}U$. Then for all $\varphi \in U$ $\theta\varphi$ is nilpotent, so $(\theta,\varphi)=\text{trace}(\theta\varphi)=0$. Hence by non-degeneracy $\theta=0$ and so U, and therefore $S(\Psi,\Psi)$, is semisimple.[15] □

II.3.8 Proposition

If $\Omega\subseteq\Gamma$ or $\Gamma\subseteq\Omega$ we have

(a) $$\dim S(\Omega,\Gamma) = \sum_{\substack{\text{all } \Psi\text{-dominant} \\ \text{weights } \lambda}} \dim V(\Omega,\Gamma,\lambda)\cdot\dim V(\Gamma,\Omega,\lambda).$$

Proof

Since $\dim S(\Omega,\Gamma) = \dim S(\Gamma,\Omega)$ it suffices to treat the case $\Omega\subseteq\Gamma$. By II.3.1 we may assume that $\text{char}(k)=0$. Let λ be a Ψ-dominant weight. Then $V(\Omega,\Gamma,\lambda)$ is an indecomposable $S(\Omega,\varnothing)$-module by II.2.11, and hence an indecomposable $S(\Psi,\Psi)$-module ($\Omega\subseteq\Gamma$ implies $S(\Omega,\varnothing)\subseteq S(\Psi,\Psi)$). Thus $V(\Omega,\Gamma,\lambda)$ is simple as an $S(\Psi,\Psi)$-module by II.3.7, and therefore certainly simple as an $S(\Omega,\Gamma)$-module. Let $P(\lambda)$ be

15 Another possible approach to this theorem is to use [Sa; (8.15)] which expresses $S(\Psi,\Psi)$ as a direct sum of tensor products of classical Schur algebras for smaller n and f.

the $S(\Gamma,\Omega)$-projective cover of the simple module $L(\Gamma,\Omega,\lambda) = V(\Omega,\Gamma,\lambda)^{\circ}$. The multiplicity of $P(\lambda)$ as a summand of $S(\Gamma,\Omega)$ is equal to the dimension of $L(\Gamma,\Omega,\lambda)$ as a module for its endomorphism algebra, i.e. to $\dim_k V(\Omega,\Gamma,\lambda)$. The module $V(\Gamma,\Omega,\lambda)$ has head $L(\Gamma,\Omega,\lambda)$, hence is a quotient of $P(\lambda)$. Thus we have:

$$\dim S(\Omega,\Gamma) = \sum_{\substack{\text{all } \Psi\text{-dominant} \\ \text{weights } \lambda}} \dim V(\Omega,\Gamma,\lambda)\cdot\dim P(\lambda)$$

$$\geq \sum_{\lambda} \dim V(\Omega,\Gamma,\lambda)\cdot\dim V(\Gamma,\Omega,\lambda)$$

Combining this with the inequality II.3.6(e) gives (a). □

Remark

Suppose $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > f$, and let λ be Ψ-dominant. The proof above shows that if $\Omega\subseteq\Gamma$ then $V(\Omega,\Gamma,\lambda)$ is simple, whilst if $\Omega\supseteq\Gamma$ it is projective.

So far we have used the characteristic independence of the dimensions of $S(\Omega,\Gamma)$ and $V(\Omega,\Gamma,\lambda)$ and the semisimplicity of $S(\Psi,\Psi)$ in characteristic zero. To extend the proof of the dimension formula to general Ω,Γ we use some combinatorial properties of the basis given in II.3.1. Put

$$j(a,b) = |\, I(a,b) \,/\, P(b)\,| = \binom{a+b-1}{b} \qquad a\in\mathbb{N}, b\in\mathbb{N}_0.$$

II.3.9 Lemma

If $a=a_1+a_2+\cdots+a_r$ with each $a_\rho\in\mathbb{N}$ then

$$j(a,b) = \sum_{(b_1,b_2,\cdots,b_r)} \prod_{\rho\in r} j(a_\rho,b_\rho),$$

where the summation is over all vectors $(b_1, b_2, \ldots, b_r)\in(\mathbb{N}_0)^r$ with $b_1+\cdots+b_r = b$.

Proof

Both sides are equal to the number of monomials of total degree b in commuting indeterminates $X_1, X_2, \cdots, X_a$. □

If $\Xi = \{\alpha, \alpha+1, \cdots, \beta\}$ is a Ψ-block, put

$$\begin{aligned} a(\Omega,\Xi) &= \#\{\ \gamma \in W_\Omega \cdot \alpha \ /\ \gamma \geq \alpha\ \}, \\ a(\Gamma,\Xi) &= \#\{\ \gamma \in W_\Gamma \cdot \alpha \ /\ \gamma \geq \alpha\ \}, \\ a(\Xi) &= \max\{\ a(\Omega,\Xi),\ a(\Gamma,\Xi)\ \}. \end{aligned}$$

Notice that $a(\Omega,\Xi), a(\Gamma,\Xi) \geq |\Xi|$, and that at most one of these inequalities is strict since otherwise $\beta+1 \in W_\Omega \cdot \alpha \cap W_\Gamma \cdot \alpha = \Xi$.

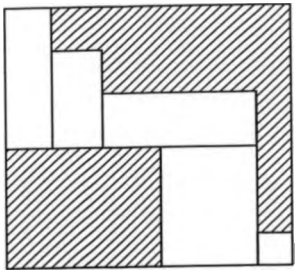
II.3.10 Lemma $$\sum_{\text{all } \Psi\text{-blocks } \Xi} a(\Xi) \cdot |\Xi| = |[\Omega\backslash\Gamma]|$$

Proof

For each Ψ-block $\Xi = \{\alpha, \alpha+1, \cdots, \beta\}$ define a subset of $[\Omega\backslash\Gamma]$ by

(a) $$A(\Xi) = \{\gamma \in W_\Omega \cdot \alpha \ /\ \gamma \geq \alpha\} \times \{\gamma' \in W_\Gamma \cdot \alpha \ /\ \gamma' \geq \alpha\}.$$

$A(\Xi)$ lies in $[\Omega\backslash\Gamma]$ since at least one of the sets appearing in the right hand side of (a) is equal to Ξ. $|A(\Xi)| = a(\Xi,\Omega) \cdot a(\Xi,\Gamma) = a(\Xi) \cdot |\Xi|$, so it is enough to show that $[\Omega\backslash\Gamma]$ is the disjoint union of the sets $A(\Xi)$. For example, if Ω and Γ are as in the diagram of II.1.2 then the sets $A(\Xi)$ are the unshaded rectangles in the following picture:



Suppose Ξ, Ξ' are distinct Ψ-blocks with $(\gamma,\gamma') \in A(\Xi)\cap A(\Xi')$. If $a(\Gamma,\Xi) = |\Xi|$, $a(\Gamma,\Xi') = |\Xi'|$ then $\gamma' \in \Xi\cap\Xi'$, a contradiction. If $a(\Gamma,\Xi) = |\Xi|$, $a(\Omega,\Xi') = |\Xi'|$ then $(\gamma,\gamma') \in [\Omega|\Psi] \cap [\Psi|\Gamma] = [\Psi|\Psi]$, and therefore $(\gamma,\gamma') \in (\Xi \times \Xi) \cap (\Xi' \times \Xi')$, which is again a contradiction. The other two cases are analogous.

Now suppose $(\gamma,\gamma') \in [\Omega|\Gamma]$. If $\gamma \geq \gamma'$ let $\Xi = \{\alpha, \alpha+1, \cdots, \beta\}$ be the Ψ-block containing γ'. Then $\gamma \in W_\Omega \cdot \alpha$, $\gamma \geq \alpha$, so $(\gamma,\gamma') \in A(\Xi)$. The case $\gamma \leq \gamma'$ is similar. □

If $\Xi = \{\alpha, \alpha+1, \cdots, \beta\}$ is a Ψ-block, and $\lambda \in \Lambda(n,f)$ put

$$\lambda(\Xi) = (\lambda_\alpha, \lambda_{\alpha+1}, \cdots, \lambda_\beta) \in \Lambda(|\Xi|).$$

We will refer to the $\lambda(\Xi)$ as the ***Ψ-components*** of λ.

For $a \in \mathbb{N}$, $b \in \mathbb{N}_0$ denote by $\Lambda^+(a,b)$ the set of dominant weights in $\Lambda(a,b)$. For $\lambda \in \Lambda^+(a,b)$, $c \in \mathbb{N}$, put

$$v(c,\lambda) = \left\{ \text{the number of } \lambda\text{-tableaux with entries in } \underline{c} \text{ which are row semi-standard and column standard.} \right.$$

II.3.11 Lemma

Suppose λ is Ψ-dominant, then

$$\dim V(\Omega,\Gamma,\lambda) = \prod_{\text{all } \Psi\text{-blocks } \Xi} v(\, a(\Omega,\Xi), \lambda(\Xi)\,).$$

Proof

dim $V(\Omega,\Gamma,\lambda)$ is (by II.3.1) the number of indices $i \in I(n,f)$ which are row semi-standard, Ψ-column standard, and such that the entries in the rows corresponding to each Ψ-block Ξ come from Ξ or from later Ψ-blocks lying in the same Ω-block. The set of such indices is in one to one correspondence with the set of vectors

$$\{\, (\, i(\Xi)\,) \,/\, \begin{array}{l}\text{for each } \Psi\text{-block } \Xi,\ i(\Xi) \in I(a(\Omega,\Xi), |\lambda(\Xi)|) \text{ is} \\ \lambda(\Xi)\text{-row semi-standard, and } \lambda(\Xi)\text{-column standard}\end{array} \,\}.$$

For each $\Xi = \{\alpha, \alpha+1, \cdots, \beta\}$, $i(\Xi)$ is obtained by restricting i to the rows corresponding to the block Ξ, shifting the domain to $\underline{r}$, where $r = |\lambda(\Xi)|$, and subtracting $\alpha-1$ from each of the entries. Counting the elements in the above set gives the required formula. □

II.3.12 Lemma

If $a,c \in \mathbb{N}$, $b \in \mathbb{N}_0$ with $a \leq c$ then

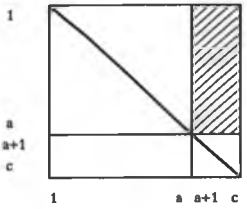
(a) $$j(ac,b) = \sum_{\lambda \in \Lambda^+(a,b)} v(c,\lambda) \cdot v(a,\lambda).$$

Proof

If $a=c$ this follows from II.3.8 and II.3.1 by taking $n=a=c$, $f=b$, $\Omega=\Gamma=\Delta$. If $b=0$ both sides are 1. Now suppose $a < c$, $b > 0$ and argue by induction on b. By II.3.9

(b) $$j(ac+(c-a)^2,b) = \sum_{\substack{b_1+b_2=b \\ b_1,b_2 \geq 0}} j(ac,b_1)\, j((c-a)^2,b_2).$$

Take $n=c$, $f=b$, $\Omega=\Delta$, $\Gamma=\Delta\backslash\{\alpha_a\}$, as in the diagram:



Using II.3.11 to expand equation II.3.8(a), we have

$$j(ac+(c-a)^2,b) = \sum_{\substack{b_1+b_2=b \\ b_1,b_2\geq 0}} \sum_{\substack{\lambda(1)\in\Lambda^+(a,b_1) \\ \lambda(2)\in\Lambda^+(c-a,b_2)}} v(a,\lambda(1))\cdot v(c,\lambda(1))\cdot v(c-a,\lambda(2))^2$$

$$\text{(c)} \qquad = \sum_{b_1,b_2} \left(\sum_{\lambda(1)} v(a,\lambda(1))\cdot v(c,\lambda(1)) \right) \left(\sum_{\lambda(2)} v(c-a,\lambda(2))^2 \right)$$

Comparing the corresponding terms in (b) and (c) for a pair (b_1,b_2), we see by using the case a=c and induction on b that these terms are equal, except possibly for the case (b,0). Since (b) and (c) are equal the latter case gives (a). □

Remark

This result is well-known to invariant theorists: Let $k[\, X_{\alpha,\gamma} \,/\, \alpha\in\underline{a},\ \gamma\in\underline{c}\,]$ be a polynomial ring in ac indeterminates. Then each side of (a) is equal to the dimension of the subspace of polynomials which are homogeneous of degree b. The left hand side is obtained by counting the basis of monomials, the right hand side by counting the basis of standard bideterminants given by the Straightening Formula. See [DKR] for example.

We can now complete the proof of II.3.4:

II.3.13 Proof of the Dimension Formula

The dimension of $S(\Omega,\Gamma)$ is $j(\#[\Omega|\Gamma], f)$, which by II.3.9 and II.3.10 is equal to

$$\sum_{(f(\Xi))} \prod_{\Xi} j(a(\Xi)\cdot|\Xi|, f(\Xi)),$$

the summation being over all vectors $(f(\Xi))$ of non-negative integers $f(\Xi)$ with $\sum_{\Xi} f(\Xi) = f$, where Ξ ranges over all Ψ-blocks. By II.3.12 this is equal to

$$\sum_{f(\Xi)} \prod_{\Xi} \sum_{\lambda(\Xi)\in\Lambda^+(|\Xi|,f(\Xi))} v(a(\Xi),\lambda(\Xi))\cdot v(|\Xi|,\lambda(\Xi)).$$

Noting that $\{a(\Xi), |\Xi|\} = \{a(\Omega,\Xi), a(\Gamma,\Xi)\}$ and assembling the component weights $\lambda(\Xi)$ into Ψ-dominant weights, we see that this is equal to

$$\sum_{\substack{\text{all } \Psi\text{-dominant}\\ \text{weights } \lambda}} \prod_{\Xi} v(a(\Omega,\Xi),\lambda(\Xi))\cdot v(a(\Gamma,\Xi),\lambda(\Xi))$$

$$= \sum_{\lambda} \dim V(\Omega,\Gamma,\lambda)\cdot\dim V(\Gamma,\Omega,\lambda),$$

by II.3.11, as required. □

II.3.14 mod $S(\Omega,\Gamma)$ as a Highest Weight Category

By II.3.4 $S(\Omega,\Gamma)$ has a bimodule filtration

$$\textbf{(a)} \qquad 0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_t = S(\Omega,\Gamma),$$

where for each $a\in[\ ]$ $J_a / J_{a-1} \cong V(\Omega,\Gamma,\lambda(a)) \otimes V(\Gamma,\Omega,\lambda(a))^J$. Fix a Ψ-dominant weight λ, let e be a primitive idempotent in $S(\Omega,\Gamma)$ corresponding to the simple module $L(\Omega,\Gamma,\lambda)$, and consider the exact functor F: mod'$S(\Omega,\Gamma) \to$ mod k, $V \mapsto Ve$ ($V \in$ mod'$S(\Omega,\Gamma)$). Applying F to (a) gives a filtration

(b) $$0 = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_u = P(\lambda),$$

of the projective cover $P(\lambda) = S(\Omega,\Gamma)e$ of $L(\Omega,\Gamma,\lambda)$, whose sections are Weyl modules. Since $L(\Omega,\Gamma,\lambda)$ appears in the head of $S(\Omega,\Gamma)^{\lambda}$ we may assume that $\xi_\lambda = e+e'$ for some idempotent e', so that for each $V \in \text{mod}^{\prime}S(\Omega,\Gamma)$ Ve is a subspace of V^{λ}. Thus by II.2.11 $F(V(\Gamma,\Omega,\mu)^{J}) = 0$ unless $\lambda \trianglelefteq_{\Gamma} \mu$, so the only Weyl modules appearing as sections of (b) are those associated with weights μ with $\lambda \trianglelefteq_{\Gamma} \mu$. Consideration of heads shows that the top section P_u / P_{u-1} is $V(\Omega,\Gamma,\lambda)$; moreover, $V(\Omega,\Gamma,\lambda)$ appears with multiplicity one as a section of (b) since the λ-weight space of $V(\Gamma,\Omega,\lambda)^{J}$ has dimension one. Exchanging the roles of Ω and Γ and applying contravariant duality we deduce using II.2.12 the following

Theorem

The injective envelope $I(\lambda)$ of $L(\Omega,\Gamma,\lambda)$ has a filtration

$$0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_v = I(\lambda),$$

where $I_1 \cong D(\Omega,\Gamma,\lambda)$, and for $v \geq i > 1$ $I_i / I_{i-1} \cong D(\Omega,\Gamma,\mu)$ for some weight μ with $\lambda \trianglelefteq_{\Omega} \mu$. The socle of $D(\Omega,\Gamma,\lambda)$ is $L(\Omega,\Gamma,\lambda)$, and if $L(\Omega,\Gamma,\mu)$ occurs as a composition factor of $D(\Omega,\Gamma,\lambda) / L(\Omega,\Gamma,\lambda)$ then $\mu \trianglelefteq_{\Gamma} \lambda$. Thus mod $S(\Omega,\Gamma)$ is a highest weight category in the sense of [CPS2], where the ordering on the simples is that given by the dominance order on their highest weights. □

Remark

In general this is not the only way to make $\text{mod}S(\Omega,\Gamma)$ into a highest weight category. Let w be the longest element of W. There is a k-algebra isomorphism:

$$\psi: S(\Omega,\Gamma) \to wS(\Omega,\Gamma)w^{-1} = S(\widehat{\Gamma},\widehat{\Omega})$$
$$\xi \mapsto w\xi w^{-1},$$

where $\widehat{\Gamma} = -w\Gamma$ and $\widehat{\Omega} = -w\Omega$. ($[\widehat{\Gamma}|\widehat{\Omega}]$ is obtained from $[\Omega|\Gamma]$ by a 180° rotation.) Let $F: \text{mod}S(\widehat{\Gamma},\widehat{\Omega}) \to \text{mod}S(\Omega,\Gamma)$ be the category isomorphism induced by ψ. We get

another highest weight category structure on $\mathrm{mod}S(\Omega,\Gamma)$ by applying F to the one exhibited in the above theorem for $\mathrm{mod}S(\bar{\Gamma},\bar{\Omega})$. One checks, using remark (iii) following II.2.11 and the fact that the weights of $F(L(\bar{\Gamma},\bar{\Omega},\lambda))$ are obtained by applying w to those of $L(\bar{\Gamma},\bar{\Omega},\lambda)$, that

$$F(L(\bar{\Gamma},\bar{\Omega},\lambda)) \cong L(\Omega,\Gamma,ww'\lambda),$$

where w' is the longest element of W_Ψ. The partial order on the Ψ-dominant weights which defines this new highest weight category structure on $\mathrm{mod}S(\Omega,\Gamma)$ is thus

$$\lambda \leq \mu \quad \text{iff} \quad w'w\lambda \trianglelefteq w'w\mu.$$

For example, if $\Omega = \Gamma = \Delta$ we get the usual dominance order; if $\Omega = \Delta$, $\Gamma = \emptyset$ we get the reverse of the dominance order. In general the set of modules $\{F(V(\bar{\Gamma},\bar{\Omega},\lambda))\}$ is different from the set $\{V(\Omega,\Gamma,\lambda)\}$. For example if $\Omega \supseteq \Gamma$ and $\mathrm{char}(k) = 0$ the former are simple, the latter projective. (See the remark following II.3.8.)

It is evident from the filtration (b) above that $V(\Omega,\Gamma,\lambda)$ can be characterized as the largest quotient module of the projective cover $P(\lambda)$ of $L(\Omega,\Gamma,\lambda)$ all of whose composition factors $L(\Omega,\Gamma,\mu)$ satisfy $\mu \trianglelefteq \lambda$. (This also follows from II.2.13.) Thus $D(\Omega,\Gamma,\lambda)$ and $V(\Omega,\Gamma,\lambda)$ are the modules denoted $A(\lambda)$ and $V(\lambda)$ in [CPS2], when $\mathrm{mod}S(\Omega,\Gamma)$ is considered as a highest weight category as in the above theorem. By the remark at the start of the proof of [CPS2; (3.11)] we have:

II.3.15 Theorem
If λ and μ are Ψ-dominant weights then

$$\mathrm{Ext}^i_{S(\Omega,\Gamma)}(V(\Omega,\Gamma,\lambda),\, D(\Omega,\Gamma,\mu)) = 0 \quad \forall\, i > 0. \;\square$$

If $U \leq V$ are k-algebras we will write Ind_U^V for the left exact induction functor $\mathrm{Hom}_U(V, \ )$: mod U → mod V. Note that Ind_U^V takes injective U-modules to injective V-modules. We are interested in the right $\mathrm{Ind}_{S(\Omega,\emptyset)}^{S(\Omega,\Gamma)}$-acyclicity or otherwise of the simple $S(\Omega,\emptyset)$-modules $k(\mu)$, for Ψ-dominant weights μ .We record the following observation:

II.3.16 Proposition

Let μ be a Ψ-dominant weight. Then $k(\mu) \in \mathrm{mod}S(\Omega,\emptyset)$ is right $\mathrm{Ind}_{S(\Omega,\emptyset)}^{S(\Omega,\Gamma)}$-acyclic iff

$$\mathrm{Ext}^i_{S(\Omega,\emptyset)}(V(\Omega,\Gamma,\lambda), k(\mu)) = 0 \quad \forall\, i > 0, \ \forall\ \Psi\text{-dominant weights } \lambda.$$

Proof

$R^i\mathrm{Ind}_{S(\Omega,\emptyset)}^{S(\Omega,\Gamma)}(k(\mu)) = \mathrm{Ext}^i_{S(\Omega,\emptyset)}(S(\Omega,\Gamma), k(\mu))$, so 'if' follows from the filtration in II.3.4 and the cohomology long exact sequence. Now suppose $k(\mu)$ is $\mathrm{Ind}_{S(\Omega,\emptyset)}^{S(\Omega,\Gamma)}$-acyclic, and take an injective resolution $k(\mu) \to I$ in mod $S(\Omega,\emptyset)$.

$$\mathrm{Ext}^i_{S(\Omega,\emptyset)}(V(\Omega,\Gamma,\lambda), k(\mu)) = H^i(\ \mathrm{Hom}_{S(\Omega,\emptyset)}(V(\Omega,\Gamma,\lambda), I)\ ),$$

and by the adjointness of restriction and induction this is isomorphic to

$$H^i(\ \mathrm{Hom}_{S(\Omega,\Gamma)}(V(\Omega,\Gamma,\lambda), \mathrm{Hom}_{S(\Omega,\emptyset)}(S(\Omega,\Gamma), I))\ ).$$

Acyclicity implies that $\mathrm{Hom}_{S(\Omega,\emptyset)}(S(\Omega,\Gamma), I))$ is an injective resolution of $D(\Omega,\Gamma,\mu)$ in mod $S(\Omega,\Gamma)$, so this is isomorphic to

$$\mathrm{Ext}^i_{S(\Omega,\Gamma)}(V(\Omega,\Gamma,\lambda), D(\Omega,\Gamma,\mu))$$

which is zero if $i > 0$, by II.3.15. □

III: Resolutions Involving a Single Root

Motivated by II.3.16 we now begin an analysis of $V(\Omega,\Gamma,\lambda)$ as an $S(\Omega,\varnothing)$-module, and specifically as a quotient of $S(\Omega,\varnothing)^\lambda$. In §1 we prove a technical result on the product $\xi_{i,j}\cdot\xi_{j,l}$ of two basis elements of the Schur algebra, when j has a certain restricted form. In §2 we define submodules M_α of $S(\Omega,\varnothing)^\lambda$ which will feature in the description of the kernel of the projection map $h: S(\Omega,\varnothing)^\lambda \to V(\Omega,\Gamma,\lambda)$, and use the results of §1 to construct a basis of M_α. In §3 we show by dimension comparison that ker h is the sum of the M_α for $\alpha\in\Psi$, and hence complete the proof of II.3.1. We deduce some further consequences for $V(\Omega,\Gamma,\lambda)$, in particular we prove a vanishing result which can be interpreted as a special case of (A'). In §4 we give an explicit $S(\Omega,\varnothing)$-projective resolution of $S(\Omega,\varnothing)^\lambda / M_\alpha$, which we use to prove (C) for a restricted set of weights, and (A') and (B') when $|\Psi| \le 1$.

In fact we will work in greater generality than we have indicated here, considering $V(\Omega,\Gamma,\lambda)$ as a quotient not just of $S(\Omega,\varnothing)^\lambda$ but also of $S(\Omega,\Gamma')^\lambda$ for subsets $\Gamma'\subseteq\Gamma$.

§1 Multiplication of Basis Elements

III.1.1 Definition

The following notation and definitions will be in force throughout §1. Take $i,l\in I(n,f)$ and suppose that for each $a\in\underline{n}$ we have a partition of $R_a(l)$:

$$R_a(l) = X_a \cup Y_a.$$

Put $X_{n+1} = Y_0 = \varnothing$. Suppose that Z is a fixed subset of $\underline{n+1}$ with $|Z| \le n$ and $Z \supseteq \{ a \in \underline{n+1} \,/\, X_a \cup Y_{a-1} \neq \varnothing \}$, and that we have an injection $t: Z \to \underline{n}$. Define $j\in I(n,f)$ by

$$j_\varphi = t(a) \quad \text{if } \varphi \in X_a \cup Y_{a-1}.$$

Example

Take l is as in the example of I.1.5. Then if $Z = \underline{n}$, and ι is the identity map, $\mathbb{T}_j$ might look like

III.1.2 Lemma

Suppose that

(a) $$\forall\, a \in \underline{n} \quad \varphi \in X_a, \varphi' \in Y_a \Rightarrow i_\varphi \neq i_{\varphi'}.$$

Then the coefficient of $\xi_{i,l}$ in the expansion of the product $\xi_{i,j}\cdot\xi_{j,l}$ is 1.

Proof

By the multiplication formula of I.2 the coefficient in question is the number of $s \in I(n,f)$ satisfying

(b) $$(i,j) \sim (i,s) \quad \text{and} \quad (j,l) \sim (s,l).$$

This number is certainly ≥ 1 (take $s = j$). Suppose there is some $s \neq j$ satisfying (b). Then $s \in jP_i \cap jP_l$. Choose $\pi \in P_i$ with $s = j\pi$. Choose $a \in \underline{n}$ minimal with $s_\varphi \neq j_\varphi$ for some $\varphi \in R_a(l)$; then since $s \in jP_l$ we can find $\varphi \in Y_a$ with $s_\varphi = \iota(a)$.

We claim that

$$\exists\, r \geq 1 \text{ such that } \pi^r\varphi \in X_a.$$

This will lead to the required contradiction, for $\pi \in P_i$ implies that $i_\varphi = i_{\pi^r\varphi}$, which is impossible by (a), since $\varphi \in Y_a$ and $\pi^r\varphi \in X_a$.

Proof of the claim: We have $j_{\pi\varphi} = s_\varphi = \iota(a)$, so $\pi\varphi \in X_a \cup Y_{a-1}$ by definition

of j. If $\pi\varphi \in X_a$ we can take r = 1. Otherwise choose $r \geq 2$ such that

$$\pi^{r-1}\varphi \in Y_{a-1} \quad \text{and} \quad \pi^r\varphi \notin Y_{a-1}.$$

Then

$$j_{\pi^r\varphi} = s_{\pi^{r-1}\varphi} \cdot j_{\pi^{r-1}\varphi} = \iota(a)$$

by minimality of a, since $\pi^{r-1}\varphi \in R_{a-1}(I)$. Hence

$$\pi^r\varphi \in (X_a \cup Y_{a-1}) \setminus Y_{a-1} = X_a,$$

as required. □

III.1.3 Lemma

Suppose that

(a) $$\forall\, a \in \underline{n} \quad \varphi \in X_a, \varphi' \in Y_{a-1} \Rightarrow i_\varphi \leq i_{\varphi'},$$

resp.

(b) $$\forall\, a \in \underline{n} \quad \varphi \in X_a, \varphi' \in Y_{a-1} \Rightarrow i_\varphi \geq i_{\varphi'}.$$

If $\xi_{i',J}$ appears with non-zero coefficient in the expansion of the product $\xi_{i,J}\cdot\xi_{j,J}$ then

$$\xi_{i',J} = \xi_{i,J} \text{ or } i' <_I i \text{ (resp. } i' >_I i).$$

Proof

We prove this in the case where (a) holds, the other case being analogous. Suppose that $\xi_{i',J} \neq \xi_{i,J}$ appears with non-zero coefficient. Then we may assume that $i' \in iP_J$. For $a \in \underline{n}$, let $\alpha(a)$, $\alpha(a,X)$, $\alpha(a,Y)$ (resp. $\beta(a)$, $\beta(a,X)$, $\beta(a,Y)$) be the weights of i (resp. i') restricted to the subsets $R_a(I)$, X_a, Y_a. Then $\forall\, a \in \underline{n}$

(c)
$$\begin{cases} \alpha(a) = \alpha(a,X) + \alpha(a,Y) \\ \beta(a) = \beta(a,X) + \beta(a,Y) \\ \alpha(a,X) + \alpha(a-1,Y) = \beta(a,X) + \beta(a-1,Y) \end{cases}$$

Choose $a \in \underline{n}$ minimal with $\alpha(a) \neq \beta(a)$. The minimality of a, together with (c) implies that $a < n$ and $\alpha(a,X) = \beta(a,X)$, so it suffices to show

(d) $$\alpha(a,Y) <_{lex} \beta(a,Y).$$

Choose $\pi \in P_j$ with $i' = i\pi$. Then π maps the subset $Y_a \cup X_{a+1}$ into itself, and we have disjoint unions:

$$Y_a = (Y_a \cap \pi Y_a) \dot\cup (Y_a \setminus \pi Y_a),$$
$$\pi Y_a = (Y_a \cap \pi Y_a) \dot\cup (\pi Y_a \cap X_{a+1}).$$

If $\varphi \in \pi Y_a \cap X_{a+1}$, and $\varphi' \in Y_a \setminus \pi Y_a$ then $i_\varphi \leq i_{\varphi'}$ by (a), so since $|Y_a \setminus \pi Y_a| = |\pi Y_a \cap X_{a+1}|$, we have

$$wt(i\,|_{Y_a \setminus \pi Y_a}) \leq_{lex} wt(i\,|_{\pi Y_a \cap X_{a+1}}).$$

Therefore

$$\alpha(a,Y) = wt(i\,|_{Y_a}) \leq_{lex} wt(i\,|_{\pi Y_a}) = wt(i'\,|_{Y_a}) = \beta(a,Y).$$

We cannot have equality here by choice of a, so (d) holds as required. □

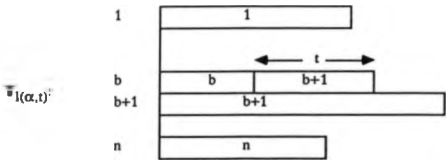
§2 The Module $M_{\alpha,r}$

Fix a weight $\lambda \in \Lambda(n,f)$. We do not for the moment assume any dominance condition on λ. Let I be the canonical index of weight λ, and use the notation of I.1.5 for this choice of λ.

III.2.1 Definitions

Fix a simple root $\alpha = \alpha_b \in \Omega$. For $t \in \{0, 1, \cdots, \lambda_b\}$ let $I(\alpha,t)$ be the index

obtained by replacing the t rightmost b's in the b^{th} row of l by b+1's, so that its associated tableau looks like



The weight of l(α,t) is λ−tα. Define $q \in \mathbb{N}$ by

$$q = \max\{\lambda_b, \lambda_{b+1}\} - \lambda_{b+1} = \begin{cases} \lambda_b - \lambda_{b+1} & \text{if } \lambda_b \geq \lambda_{b+1} \\ 0 & \text{otherwise} \end{cases}$$

Take an integer $r \in \{1, \cdots, \lambda_b - q\}$. We define the module $M_{\alpha,r} = M_{\alpha,r}(\Omega,\Gamma,\lambda)$ to be the $S(\Omega,\Gamma)$-submodule of $S(\Omega,\Gamma)^{\lambda}$ generated by the elements $\xi_{l(\alpha,t),l}$ for $t \in \{q+r, q+r+1, \cdots, \lambda_b\}$. These elements lie in $S(\Omega,\Gamma)$ because $\alpha \in \Omega$. It is easy to see that $\xi_{l(\alpha,t),l}$ spans $^{\lambda-t\alpha}S(\Omega,\emptyset)^{\lambda}$. If $\lambda_b < q+r$ put $M_{\alpha,r} = 0$. Our main interest is in the module $M_{\alpha} = M_{\alpha}(\Omega,\Gamma,\lambda) = M_{\alpha,1}$. We introduce the $M_{\alpha,r}$ because they will appear later in the construction of a projective resolution of $S(\Omega,\Gamma)^{\lambda} / M_{\alpha}$, and it is convenient to treat all these modules together in a uniform manner.

We will say that a basis element $\xi_{i,l}$ is **(α,r)-*faulted*** if there exists $c \in \underline{n}$ such that

(a) $$\#\{\varphi \in R_b(l) \,/\, i_\varphi \geq c\} + \#\{\varphi \in R_{b+1}(l) \,/\, i_\varphi \leq c\} \geq r + \max\{\lambda_b, \lambda_{b+1}\}.$$

If $\lambda_b < q+r$, no $\xi_{i,l}$ is (α,r)-faulted. If $\xi_{i,l}$ is (α,r)-faulted it is also (α,r')-faulted for any r' with $1 \leq r' \leq r$. We can describe the notion of (α,r)-faultedness in another way as a generalization of the property of being {α}-non-standard:

III.2.2 Lemma

(i) Suppose $\lambda_b \geq \lambda_{b+1}$, and take i to be row semi-standard. Then $\xi_{i,l}$ is (α,r)-

faulted iff $\exists$ d with $1 \le d \le \lambda_{b+1}-r+1$ such that $\mathbb{T}_i(b,d) \ge \mathbb{T}_i(b+1,d+r-1)$.

(ii) Suppose $\lambda_b \le \lambda_{b+1}$, and take i to be reverse row semi-standard. Then $\xi_{i,J}$ is (α,r)-faulted iff $\exists$ d with $r \le d \le \lambda_b$ such that $\mathbb{T}_i(b,d) \ge \mathbb{T}_i(b+1,d-r+1)$.

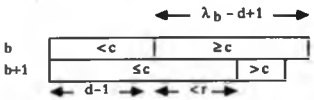
If r=1 we conclude (using I.1.6) that $\xi_{i,J}$ is $(\alpha,1)$-faulted iff it is $\{\alpha\}$-non-standard.

Proof

We will prove (i), the proof of (ii) being analogous. If III.2.1(a) holds, choose d minimal with $\mathbb{T}_i(b,d) \ge c$. Then $d+r-1 \le \lambda_{b+1}$ and $\mathbb{T}_i(b,d) \ge c \ge \mathbb{T}_i(b+1,d+r-1)$, for otherwise

$$\#\{\varphi \in R_b(I) \,/\, i_\varphi \ge c\} + \#\{\varphi \in R_{b+1}(I) \,/\, i_\varphi \le c\} \le (\lambda_b - d+1)+(d+r-2) < \lambda_b + r,$$

by row standardness of i. (See the diagram below.)



Conversely if $\mathbb{T}_i(b,d) \ge \mathbb{T}_i(b+1,d+r-1)$ we get III.2.1(a) by putting $c = \mathbb{T}_i(b,d)$. □

We will show in III.2.8 that the dimension of $M_{\alpha,r}$ is equal to the number of (α,r)-faulted basis elements $\xi_{i,J} \in S(\Omega,\Gamma)^\lambda$. For the moment we prove that it is at least this number by using the results of §1 to construct a suitable collection of linearly independent elements in $M_{\alpha,r}$.

III.2.3 Construction of a Basis for $M_{\alpha,r}$

Suppose that $\xi_{i,J} \in S(\Omega,\Gamma)^\lambda$ is (α,r)-faulted, and choose c satisfying III.2.1(a). We will find an index j with $\xi_{j,J} \in M_{\alpha,r}$, $\xi_{i,j} \in S(\Omega,\Gamma)$ and

(a) $\quad \xi_{i,j}\cdot\xi_{j,J} = \xi_{i,J}$ + a linear combination of terms $\xi_{i',J}$ with $i' <_l i$.

The collection of all these elements $\xi_{i,j}\cdot\xi_{j,l}$, one for each (α,r)-faulted $\xi_{i,l} \in S(\Omega,\Gamma)^{\lambda}$, will turn out to be a basis of $M_{\alpha,r}$ (see III.2.5 and III.2.8). We split the construction into two cases according to whether $b < c$ or $b \geq c$.

Case I: $b < c$

Define for each $a \in \underline{n}$ a partition $R_a(l) = X_a \cup Y_a$, as follows:

$$X_a = R_a(l),\ Y_a = \emptyset \qquad \text{if } a < b;$$
$$X_b = \{\varphi \in R_b(l) \,/\, i_\varphi < c\},\ Y_b = R_b(l) \setminus X_b \qquad \text{if } a = b.$$

For $a > b$, define X_a and Y_a inductively by

$$X_a = \{\varphi \in R_a(l) \,/\, i_\varphi \leq i_{\varphi'},\ \forall\, \varphi' \in Y_{a-1}\},\ Y_a = R_a(l) \setminus X_a.$$

We claim that $Y_n = \emptyset$. If not, then $Y_b, Y_{b+1}, \cdots, Y_n \neq \emptyset$. Let $y_a = \min\{i_\varphi \,/\, \varphi \in Y_a\}$ for $a = b, \cdots, n$. Then $c \leq y_b < y_{b+1} < \cdots < y_n \leq n$, so $b \geq c$, a contradiction. As in III.1.1, we put $Y_0 = \emptyset$ and define an index j by

$$j_\varphi = a \text{ if } \varphi \in X_a \cup Y_{a-1}.$$

In the notation of III.1.1, $Z = \underline{n}$ and $\iota: Z \to \underline{n}$ is the identity map. By construction the hypotheses III.1.2(a) and III.1.3(a) hold so we have (a) above. Notice that by choice of c we have

(b) $$\#R_{b+1,b}(j,l) + \#R_{b+1,b+1}(j,l) \geq r + \max\{\lambda_b, \lambda_{b+1}\}.$$

We now show that $\xi_{i,j}$ and $\xi_{j,l}$ both lie in $S(\Omega,\Gamma)$. Take $a \in \underline{n}$, $\varphi \in R_a(l)$. We must show that (i_φ, j_φ), $(j_\varphi, l_\varphi) \in [\Omega|\Gamma]$, given that $(i_\varphi, l_\varphi) \in [\Omega|\Gamma]$. If $\varphi \in X_a$ then $j_\varphi = l_\varphi$, so this is certainly the case. Otherwise $\varphi \in Y_a$ with $a \geq b$, and

$$i_\varphi \geq y_a \geq c + a - b \geq a + 1 = j_\varphi > a = l_\varphi,$$

giving the required result.

Case II: $b \geq c$

Define for each $a \in \underline{n}$ a partition $R_a(l) = X_a \cup Y_a$ by

$$X_a = \emptyset,\ Y_a = R_a(l) \qquad \text{if } a > b+1;$$
$$X_{b+1} = \{\varphi \in R_{b+1}(l) \,/\, i_\varphi \leq c\},\ Y_{b+1} = R_{b+1}(l) \setminus X_{b+1} \qquad \text{if } a = b+1.$$

For $a \leq b$, define X_a and Y_a inductively by

$$Y_a = \{\varphi \in R_a(l) \,/\, i_\varphi \geq i_{\varphi'},\ \forall\, \varphi' \in X_{a+1}\},\ X_a = R_a(l) \setminus Y_a.$$

$X_1 = \emptyset$, for if not $X_1, X_2, \cdots, X_{b+1} \neq \emptyset$. Let $x_a = \max\{i_\varphi \,/\, \varphi \in X_a\}$ for $a = 1, \cdots, b+1$. Then $1 \leq x_1 < x_2 < \cdots < x_{b+1} \leq c$, so $c > b$, a contradiction. We put $X_{n+1} = \emptyset$ and define an index j by

$$j_\varphi = a \text{ if } \varphi \in Y_a \cup X_{a+1}.$$

In the notation of III.1.1, $Z = \{2, 3, \cdots, n+1\}$ and $\iota : Z \to \underline{n}$ is given by subtracting 1. Again the hypotheses III.1.2(a) and III.1.3(a) hold giving (a) above, and

(c) $$\#R_{b,b}(j,l) + \#R_{b,b+1}(j,l) \geq r + \max\{\lambda_b, \lambda_{b+1}\}.$$

To show that $\xi_{i,j}$ and $\xi_{j,l}$ lie in $S(\Omega,\Gamma)$, take $a \in \underline{n}$, $\varphi \in R_a(l)$. If $\varphi \in Y_a$ then $j_\varphi = l_\varphi$. Otherwise $\varphi \in X_a$ with $a \leq b+1$, and

$$i_\varphi \leq x_a \leq c - b + a - 1 \leq a - 1 = j_\varphi < a = l_\varphi.$$

In either case $(i_\varphi, j_\varphi), (j_\varphi, l_\varphi) \in [\Omega|\Gamma]$, as required.

To complete the construction we must prove the following:

III.2.4 Lemma

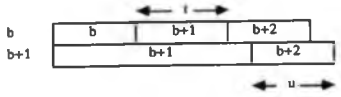
The element $\xi_{j,l}$ defined above lies in $M_{\alpha,r}$.

Proof

Case I: $b < c$

Rows b and b+1 of the index j can be rearranged to take the form

(a)



where, by III.2.3(b)

$$t-u \geq \max(\lambda_b, \lambda_{b+1}) - \lambda_{b+1} + r = q+r.$$

(In fact there are no b+2's in the b^{th} row of j, but we need this more general format in the proof.) We will show that if $i \in I(n,f)$ is such that $\xi_{i,j} \in S(\Omega,\Gamma)$, and rows b and b+1 have the form (a) above (up to rearrangement), with $t-u \geq q+r$, then $\xi_{i,j} \in M_{\alpha,r}$. (Caution: the i and j appearing in this proof from here onwards are not the same as those in III.2.3. We are over-using these symbols to maintain notational compatibility with §1.)

For each $a \in \underline{n}$ define a partition of $R_a(l)$ by

$$X_a = \begin{cases} R_a(l) & \text{if } a \neq b \\ R_b(l) \setminus R_{b+1,b}(i,j) & \text{if } a = b \end{cases}$$

$$Y_a = \begin{cases} \emptyset & \text{if } a \neq b \\ R_{b+1,b}(i,j) & \text{if } a = b. \end{cases}$$

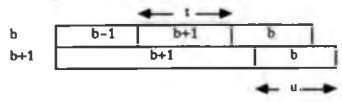
The index $j \in I(n,f)$ defined by $j_\varphi = a$ if $\varphi \in X_a \cup Y_{a-1}$ can be obtained from $l(\alpha,t)$ by reordering the b^{th} row, and $t \geq q+r$ by assumption, so $\xi_{j,j} \in M_{\alpha,r}$. For any $\varphi \in f$, j_φ is either i_φ or l_φ, so $\xi_{i,j} \in S(\Omega,\Gamma)$. Hypotheses III.1.2(a) and III.1.3(b) hold, giving

$$\xi_{i,j}\xi_{j,j} = \xi_{i,j} + \text{a linear combination of terms } \xi_{i',j} \text{ with } i' >_j i .$$

If $\xi_{i',J}$ appears with non-zero coefficient on the right then i' can be assumed to be obtained from i by permuting the entries of i within the sets $R_a(j)$. Rows b and b+1 of i' can thus be rearranged into the form (a) with t, u replaced by t-v, u-v for some $v \in \{0\} \cup \underline{u}$. By induction on the order $>_l$ we may thus assume that all such $\xi_{i',J}$ lie in $M_{\alpha,r}$, so $\xi_{i,J} \in M_{\alpha,r}$.

Case II: $b \geq c$

The assumption that $\xi_{i,J}$ is (α,r)-faulted implies in particular that there is some $\varphi \in R_{b+1}(I)$ with $i_\varphi \leq c \leq b < b+1 = l_\varphi$, so $i_\varphi \sim_\Gamma l_\varphi$ and $\alpha \in \Gamma$. Thus $s_\alpha \in S(\Omega,\Gamma)$, and it is enough to show that $s_\alpha \cdot \xi_{j,J} = \xi_{s_\alpha j,J}$ lies in $M_{\alpha,r}$. Rows b and b+1 of the index $s_\alpha \cdot j$ can be reordered to take the form



where by III.2.3(c), $t-u \geq q+r$. As in case I, any basis element $\xi_{i,J}$ which has this form lies in $M_{\alpha,r}$, the proof being similar to case I, except that condition III.1.3(a) holds instead of III.1.3(b) and so we use induction on the order $<_l$ instead of the order $>_l$. □

We can now prove:

III.2.5 Proposition

With notation and assumptions as above, $M_{\alpha,r}$ has dimension at least the number of (α,r)-faulted basis elements $\xi_{i,J}$ in $S(\Omega,\Gamma)^\lambda$.

Proof

For each such basis elements $\xi_{i,J}$ we have constructed an element $\xi_{i,J} \cdot \xi_{j,J}$ of $M_{\alpha,r}$ [16], which has $\xi_{i,J}$ as its leading term when the standard basis elements in

[16] This element is not uniquely determined since it depends upon the choice of c in the construction. For each basis element $\xi_{i,J}$ we fix some particular c.

$S(\Omega,\Gamma)^\lambda$ are totally ordered by the relation $<_1$. These elements of $M_{\alpha,r}$ are thus linearly independent. □

Remarks

(i) Take a subset $\Theta\subset\Omega$. If λ is Θ-dominant and $\xi_{i,j}$ is Θ-non-standard then it is $(\alpha,1)$-faulted for some $\alpha\in\Theta$ by I.1.6 and III.2.2. Fix some such choice of α and apply the above construction to produce an element $\xi_{i,j}\cdot\xi_{j,l}\in M_\alpha$. Then the set of all such $\xi_{i,j}\cdot\xi_{j,l}$ is linearly independent in $\sum_{\alpha\in\Theta} M_\alpha$. It can in fact be shown that this set is a basis for $\sum_{\alpha\in\Theta} M_\alpha$, but we will not pursue this here.

(ii) We can regard the construction as a 'straightening' process: it defines an algorithm which, given a Θ-non-standard basis element $\xi_{i,j}$, expresses it modulo the submodule $\sum_{\alpha\in\Theta} M_\alpha$ as a linear combination of Θ-standard basis elements.

The following result follows readily from the multiplication formula in I.2:

III.2.6 Lemma

If $\lambda_b \geq u \geq v \geq w \geq 0$ then

$$\xi_{I(\alpha,u),I(\alpha,v)}\cdot\xi_{I(\alpha,v),I(\alpha,w)} = \binom{u-w}{v-w}\,\xi_{I(\alpha,u),I(\alpha,w)}.\ \square$$

The proof of the following lemma is an easy exercise.

III.2.7 Lemma

(i) If $b \geq b'$ and $b \sim_\Omega b'$ then $(a,b) \in [\Omega|\Gamma]$ implies $(a,b') \in [\Omega|\Gamma]$.

(ii) If $a \leq a' \leq a''$ and $b \geq b' \geq b''$ then (a,b), $(a'',b'') \in [\Omega|\Gamma]$ implies $(a',b') \in [\Omega|\Gamma]$. □

III.2.8 Theorem (Basis of $M_{\alpha,r}$)

With notation and assumptions as above,

$$\dim M_{\alpha,r} = \dim S(\Omega,\Gamma)^\lambda - (q+r)\alpha \quad [17]$$

$$= \text{the number of } (\alpha,r)\text{-faulted basis elements in } S(\Omega,\Gamma)^{\lambda}.$$

Thus the elements $\xi_{i,j}\cdot\xi_{j,1}$ of III.2.3(a) form a basis of $M_{\alpha,r}$.

In characteristic zero $M_{\alpha,r}$ is generated by $\xi_{1(\alpha,q+r),1}$, and

$$M_{\alpha,r} \cong S(\Omega,\Gamma)^{\lambda-(q+r)\alpha}.$$

Notice that if λ is $\{\alpha\}$-dominant $\lambda-(q+1)\alpha = s_{\alpha}\circ\lambda$. (This observation will be important in the next chapter.)

Proof

Let X be the integer matrix whose rows give the coefficients of the various products $\xi_{i,j}\cdot\xi_{1(\alpha,t),1}$ $(\xi_{i,j}\in S(\Omega,\Gamma),\ t\in\{q+r, q+r+1, \cdots, \lambda_b\})$ when expressed as linear combinations of the basis $\{\xi_{i,j}\}$ of $S(\Omega,\Gamma)$. Since $\dim M_{\alpha,r}$ is the rank of X (when the entries of X are regarded as elements of k), the dimension of $M_{\alpha,r}$ in positive characteristic is no greater than its dimension in characteristic zero. By III.2.6 $M_{\alpha,r}$ is generated in characteristic zero by $\xi_{1(\alpha,q+r),1}$ so $\dim M_{\alpha,r} \le \dim S(\Omega,\Gamma)^{\lambda-(q+r)\alpha}$. In view of III.2.5 the theorem will follow once we establish that

$$\dim S(\Omega,\Gamma)^{\lambda-(q+r)\alpha} = \text{the number of } (\alpha,r)\text{-faulted basis elements in } S(\Omega,\Gamma)^{\lambda}.$$

If $\lambda_b < q+r$, both of the numbers in question are zero, so suppose $\lambda_b \ge q+r$. Put $\mu = \lambda-(q+r)\alpha$ and let m be the canonical index of weight μ. We will define a bijection

$$\psi: \{\xi_{i,m}\in S(\Omega,\Gamma)^{\mu}\} \to \{\xi_{j,1}\in S(\Omega,\Gamma)^{\lambda} \,/\, \xi_{j,1} \text{ is } (\alpha,r)\text{-faulted}\}.$$

We do this in two cases depending upon the sign of $\lambda_b - \lambda_{b+1}$.

Case I: $\lambda_b \ge \lambda_{b+1}$

Take $\xi_{i,m}$ with i μ-row semi-standard. Write the b and b+1 μ-rows of i as

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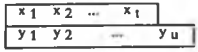
17 If V is an $S(\emptyset,\emptyset)$-module, and μ a non-polynomial weight we interpret the weight space ${}^{\mu}V$ as being zero. This is consistent with the algebraic group definition of weight space.

(a)

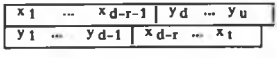


where $t = \mu_b = \lambda_{b+1} - r$ and $u = \mu_{b+1} = \lambda_b + r$. Put

$$d = \max\{\, c \in \{r+1, r+2, \dots, r+t+1\} \;/\; \text{if } c > r+1 \text{ then } y_c \geq x_{c-r-1} \,\},$$

and let $\psi(\xi_{i,m}) = \xi_{j,l}$, where j is the index whose a^{th} λ-row is the a^{th} μ-row of i if $a \notin \{b, b+1\}$ and whose b and b+1 λ-rows are

(b)



Notice that j is λ-row semi-standard. j is (α,r)-faulted by III.2.2 since

(c) $$\mathbb{T}_j^{\lambda}(b,d-r) = y_d \geq y_{d-1} = \mathbb{T}_j^{\lambda}(b+1,d-1),$$

and d-r is the rightmost column with this property, for

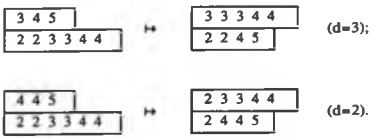
(d) $$\mathbb{T}_j^{\lambda}(b,c-r) = y_c < x_{c-r-1} = \mathbb{T}_j^{\lambda}(b+1,c-1) \quad \forall c \text{ with } \lambda_{b+1}+1 \geq c > d,$$

by maximality of d. We can therefore recover d and hence i from $\xi_{j,l}$, so the map ψ is injective. If j is λ-row semi-standard and $\xi_{j,l}$ is (α,r)-faulted, we can write its b and b+1 rows as in (b) so that (c) and (d) hold, so $\xi_{j,l} \in \text{im}\psi$.

It remains to show that $\xi_{j,l} \in S(\Omega,\Gamma)$ iff $\xi_{i,m} \in S(\Omega,\Gamma)$. i is obtained from j, and vice-versa, by exchanging a block of entries at the end of row b with a block of entries at the end of row b+1. This is done in such a way that row semi-standardness is preserved and some non-empty part of row b+1 remains unaltered. By III.2.7(i), $(y,b+1) \in [\Omega|\Gamma]$ and $\alpha \in \Omega$ implies that $(y,b) \in [\Omega|\Gamma]$. On the other hand, if $(x,b) \in [\Omega|\Gamma]$ and $x \geq y$ for some y with $(y,b+1) \in [\Omega|\Gamma]$, then $(x,b+1) \in [\Omega|\Gamma]$ by III.2.7(ii).

Examples

Take r = 1.

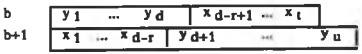


Case II: $\lambda_b < \lambda_{b+1}$

Take $\xi_{i,m}$ with i reverse μ-row semi-standard, and write the b and b+1 μ-rows of i as in (a), but where now $x_1 \geq x_2 \geq \cdots \geq x_t$, $y_1 \geq y_2 \geq \cdots \geq y_u$, $t = \mu_b = \lambda_b - r$ and $u = \mu_{b+1} = \lambda_{b+1}+r$. Put

$$d = \min\{\, c \in \{r, r+1, \cdots, r+t\} \;/\; \text{if } c < r+t \text{ then } y_c \geq x_{c-r+1} \,\}.$$

and let $\psi(\xi_{i,m}) = \xi_{j,l}$, where j is the index whose a^{th} λ-row is the a^{th} μ-row of i if $a \notin \{b, b+1\}$ and whose b and b+1 λ-rows are



j is reverse λ-row semi-standard, and (α,r)-faulted since

$$T_j^{\lambda}(b,d) = y_d \geq y_{d+1} = T_j^{\lambda}(b+1,d-r+1).$$

d is the leftmost column with this property, for if $r \leq c < d$

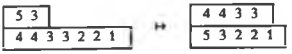
$$T_j^{\lambda}(b,c) = y_c < x_{c-r+1} = T_j^{\lambda}(b+1,c-r+1).$$

We conclude as before that ψ is an injection whose image consists of all (α,r)-faulted

$\xi_{j,1}$. That $\xi_{j,1} \in S(\Omega,\Gamma)$ iff $\xi_{i,m} \in S(\Omega,\Gamma)$ follows much as in case I. □

Example

Take r=2



(d=3).

§3 The Kernel of the Map $S(\Omega,\varnothing)^{\lambda} \to V(\Omega,\Gamma,\lambda)$

III.3.1 Theorem

Suppose λ is a Ψ-dominant weight, and $\Gamma' \subseteq \Psi$. Then the kernel of the epimorphism

$$h: S(\Omega,\Gamma')^{\lambda} \to V(\Omega,\Gamma,\lambda)$$
$$\xi \mapsto \xi \otimes 1$$

is the sum over all $\alpha \in \Psi$ of the submodules $M_\alpha = M_\alpha(\Omega,\Gamma',\lambda)$. The images of the Ψ-standard $\xi_{i,1}$ under this map form a basis of $V(\Omega,\Gamma,\lambda)$. Taking $\Gamma' = \varnothing$ gives II.3.1.

Proof

Take $\alpha = \alpha_b \in \Psi$. Then M_α is generated by weight vectors for weights of the form $\lambda - t\alpha$, where $t \geq \lambda_b - \lambda_{b+1} + 1$ (see III.2.1). Put $\mu = s_\alpha(\lambda - t\alpha)$. Then

$$\mu_1 + \cdots + \mu_b = \lambda_1 + \cdots + \lambda_{b-1} + \lambda_{b+1} + t$$

$$\geq \lambda_1 + \cdots + \lambda_b + 1,$$

so $s_\alpha(\lambda - t\alpha) \not\trianglelefteq \lambda$. Since the weights of $V(\Omega,\Gamma,\lambda)$ are permuted by W_Ψ and every weight of $V(\Omega,\Gamma,\lambda)$ is dominated by λ, $\lambda - t\alpha$ is not a weight of $V(\Omega,\Gamma,\lambda)$. Hence M_α lies in ker h.

We now show that if $\xi_{i,j} \in S(\Omega,\Gamma')^\lambda$ is Ψ-standard then $\xi_{i,j} \in S(\Omega,\emptyset)^\lambda$. Suppose $\xi_{i,j} \in S(\Omega,\Gamma')^\lambda \setminus S(\Omega,\emptyset)^\lambda$, and choose $b \in \underline{n}$ minimal such that there is some $\varphi \in R_{b+1}(l)$ with $i_\varphi < b+1$. Then $\alpha_b \in \Gamma' \subseteq \Psi$ so $\lambda_b \geq \lambda_{b+1}$ by Ψ-dominance of λ, and $i_{\varphi'} \geq b \geq i_\varphi$ for all $\varphi' \in R_b(l)$ by minimality of b. Thus $\xi_{i,j}$ is Ψ-non-standard. We know by II.2.10 that the image of h has dimension at least the number of Ψ-standard $\xi_{i,j} \in S(\Omega,\emptyset)^\lambda$, and this is the same as the number of Ψ-standard $\xi_{i,j} \in S(\Omega,\Gamma')^\lambda$. The dimension of $\ker h$ is thus at most the number of Ψ-non-standard $\xi_{i,j} \in S(\Omega,\Gamma')^\lambda$. By remark (i) following III.2.5, $\dim \sum_{\alpha \in \Psi} M_\alpha$ is at least this number, so we have $\ker h = \sum_{\alpha \in \Psi} M_\alpha$. Furthermore, the images of the Ψ-standard $\xi_{i,j} \in S(\Omega,\Gamma')^\lambda$ are independent by II.2.10 and we have demonstrated that their number is the dimension of $V(\Omega,\Gamma,\lambda)$, so they form a basis. □

Remarks

(i) We could have arrived directly at the proof of II.3.1 with somewhat less work: we only needed to consider $M_{\alpha,\Gamma}(\Omega,\Gamma,\lambda)$ for $r=1$, λ $\{\alpha\}$-dominant, and $\Gamma=\emptyset$. The latter restriction obviates the need for case II of III.2.3 and III.2.4. We have not yet used the basis theorem III.2.8 for $M_{\alpha,\Gamma}$.

(ii) We will see in the proof of III.3.5 that in general $\sum_{\alpha \in \Psi} M_\alpha(\Omega,\Gamma,\lambda)$ is the kernel of the obvious epimorphism $S(\Omega,\Gamma)^\lambda \twoheadrightarrow S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes} V(\Omega,\Gamma,\lambda)$.

III.3.2 Corollary

Suppose λ is Ψ-dominant. Then $V(\Omega,\Gamma,\lambda) \cong V(\Omega,\Psi,\lambda)$ as $S(\Omega,\Psi)$-modules.

Proof

By III.3.1 $V(\Omega,\Gamma,\lambda)$ and $V(\Omega,\Psi,\lambda)$ are both isomorphic to the quotient of $S(\Omega,\Psi)^\lambda$ by the submodule $\sum_{\alpha \in \Psi} M_\alpha(\Omega,\Psi,\lambda)$. □

III.3.3 Theorem

If λ and μ are Ψ-dominant weights then

$$\mathrm{Ext}^1_{S(\Omega,\emptyset)}(V(\Omega,\Gamma,\lambda), k(\mu)) = 0.$$

Proof

By III.3.1 we have a short exact sequence of $S(\Omega,\emptyset)$-modules:

$$0 \to \sum_{\alpha\in\Psi} M_\alpha(\Omega,\emptyset,\lambda) \to S(\Omega,\emptyset)^\lambda \to V(\Omega,\Gamma,\lambda) \to 0.$$

For $\alpha\in\Psi$, $M_\alpha = M_\alpha(\Omega,\emptyset,\lambda)$ is generated by weight vectors for weights μ with $s_\alpha\mu \not\trianglelefteq \lambda$ (as in the proof of III.3.1). No such μ is Ψ-dominant, for otherwise $s_\alpha\mu \trianglelefteq \mu \trianglelefteq \lambda$. Thus $V(\Omega,\Gamma,\lambda)$ has a two-step projective presentation of the form

$$\coprod_c S(\Omega,\emptyset)^{\mu(c)} \to S(\Omega,\emptyset)^\lambda \to V(\Omega,\Gamma,\lambda) \to 0,$$

where for each c the weight $\mu(c)$ is not Ψ-dominant. The theorem follows. □

Remark

This theorem (together with the fact that the $S(\Omega,\emptyset)$-heads of the Weyl modules are simple) implies that if $V \in \mathrm{mod}S(\Omega,\emptyset)$ has a filtration by Weyl modules, then $\dim \mathrm{Hom}_{S(\Omega,\emptyset)}(V, k(\mu))$ counts the number of times that $V(\Omega,\Gamma,\mu)$ occurs in any such filtration. (It follows from II.3.15 that $\dim \mathrm{Hom}_{S(\Omega,\Gamma)}(V, D(\Omega,\Gamma,\mu))$ has the same interpretation – cf. [CPS2; (3.11)].)

The following corollary is a first step towards a proof of (A'):

III.3.4 Corollary

$$R^1 \mathrm{Ind}_{S(\Omega,\emptyset)}^{S(\Omega,\Gamma)} k(\mu) = 0 \quad \forall\ \Psi\text{-dominant weights } \mu.$$

Proof

By II.3.4 $S(\Omega,\Gamma)$ has a filtration as a left $S(\Omega,\Gamma)$-module with each section isomorphic to some $V(\Omega,\Gamma,\lambda)$ for a Ψ-dominant weight λ. The corollary follows

from the previous theorem by the cohomology long exact sequence. □

III.3.5 Fully Faithfulness of Restriction

It is a consequence of the fact that the quotient variety G/B^- is projective that the restriction functor from rational G-modules to rational B^--modules is fully faithful. Since mod S(G) and mod $S(B^-)$ are full subcategories of the respective categories of rational modules, the restriction functor from mod S(G) to mod $S(B^-)$ is also fully faithful. We use III.3.1 to give an 'internal' proof of this fact, which is the dimension zero case of (B').

Proposition

The restriction functor from mod $S(\Omega,\Gamma)$ to mod $S(\Omega,\emptyset)$ is fully faithful iff $\Omega \supseteq \Gamma$.

Proof

The restriction functor is fully faithful iff for all V in mod$S(\Omega,\Gamma)$ the natural epi $S(\Omega,\Gamma) \underset{S(\Omega,\Gamma)}{\otimes} V \to V$ is an isomorphism (see e.g. [M; IV (3.1)]). The natural isomorphism $S(\Omega,\Gamma) \underset{S(\Omega,\Gamma)}{\otimes} V \cong V$ shows that to prove fully faithfulness it is enough to establish that $S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes} V \to V$ is an isomorphism for $V = S(\Omega,\Gamma)$, and hence using II.3.4 and an easy induction, that it is an isomorphism for the Weyl modules.

As in the proof of III.3.3, $V(\Omega,\Gamma,\lambda)$ has an $S(\Omega,\emptyset)$-projective presentation of the form

$$\coprod_c S(\Omega,\emptyset)^{\mu(c)} \to S(\Omega,\emptyset)^\lambda \to V(\Omega,\Gamma,\lambda) \to 0,$$

where the weights $\mu(c)$ have the form $\lambda - t\alpha$ for $\alpha \in \Psi$ and certain positive integers t, and the components of the left hand map are given by multiplication by the appropriate element $\xi_{[(\alpha,t)]}$ (see III.2.1). Applying $S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes}$ gives an exact sequence:

$$\coprod_c S(\Omega,\Gamma)^{\mu(c)} \to S(\Omega,\Gamma)^\lambda \to S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes} V(\Omega,\Gamma,\lambda) \to 0.$$

and hence a short exact sequence

$$0 \to \sum_{\alpha\in\Psi} M_\alpha(\Omega,\Gamma,\lambda) \to S(\Omega,\Gamma)^\lambda \to S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes} V(\Omega,\Gamma,\lambda) \to 0.$$

If $\Omega\supseteq\Gamma$ then $\Gamma = \Psi$ so we can apply III.3.1 to deduce that $V(\Omega,\Gamma,\lambda)$ itself has a resolution of just this form, so $V(\Omega,\Gamma,\lambda) \cong S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes} V(\Omega,\Gamma,\lambda)$.

Now suppose $\Omega\not\supseteq\Gamma$ and take $\alpha_b \in \Gamma \setminus \Omega$. Put $\lambda = (0, \cdots, 0, f, 0, \cdots, 0)$ with the f occurring in position b+1. It is easy to see that $M_\beta(\Omega,\emptyset,\lambda) = M_\beta(\Omega,\Gamma,\lambda) = 0$ for all $\beta\in\Psi$ so we have isomorphisms

$$V(\Omega,\Gamma,\lambda) \cong S(\Omega,\emptyset)^\lambda$$

and

$$S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes} V(\Omega,\Gamma,\lambda) \cong S(\Omega,\Gamma)^\lambda.$$

However $S(\Omega,\emptyset)^\lambda \not\cong S(\Omega,\Gamma)^\lambda$ since $\xi_{i,j} \in S(\Omega,\Gamma)^\lambda \setminus S(\Omega,\emptyset)^\lambda$, where for all $\varphi\in f$ $i_\varphi = b$ and $l_\varphi = b+1$, so the map $S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes} V(\Omega,\Gamma,\lambda) \to V(\Omega,\Gamma,\lambda)$ is not an isomorphism. □

§4 A Projective Resolution of $S(\Omega,\Gamma)^\lambda$ / $M_{\alpha,r}$

III.4.1 A Resolution

Suppose we are given the following data:

(i) $N_1, N_2, \cdots, N_t \in \mathrm{mod}U$, for some k-algebra U and some $t \geq 2$.

(ii) For each pair (τ,σ) with $t \geq \tau > \sigma \geq 1$ a U-map

$$h_{\tau,\sigma}: N_\tau \to N_\sigma$$

satisfying conditions:

(iii) If $t \geq \tau > \sigma > \rho \geq 1$ then the composite $h_{\sigma,\rho}h_{\tau,\sigma}$ is a scalar multiple of $h_{\tau,\rho}$.

(iv) If $t \geq \sigma > \rho \geq 1$ then

$$\dim \sum_{\tau \geq \sigma} \operatorname{im} h_{\tau,\rho} = \dim N_{\sigma}.$$

Put $H_{\tau,\sigma} = k\text{-}h_{\tau,\sigma} \leq \operatorname{Hom}_U(N_\tau, N_\sigma)$. If $I=\{i_\tau > i_{\tau-1} > \cdots > i_1\}$ is a non-empty subset of t put

$$N_I = N_{i_\tau} \otimes H_{i_\tau,i_{\tau-1}} \otimes H_{i_{\tau-1},i_{\tau-2}} \otimes \cdots \otimes H_{i_2,i_1}.$$

This is a U-module isomorphic to N_{i_τ}. Suppose now that $\tau \geq 2$, and take σ with $\tau \geq \sigma \geq 2$. Put $I' = I \setminus \{i_\sigma\}$. We define a U-map $\partial(I,I'): N_I \to N_{I'}$ as follows: if $\sigma = \tau$, $\partial(I,I')$ is the map induced by evaluation of functions

$$N_{i_\tau} \otimes H_{i_\tau,i_{\tau-1}} \to N_{i_{\tau-1}}.$$

If $\sigma < \tau$, $\partial(I,I')$ is the map induced by composition of functions

$$H_{i_{\sigma+1},i_\sigma} \otimes H_{i_\sigma,i_{\sigma-1}} \to H_{i_{\sigma+1},i_{\sigma-1}}.$$

We also attach a sign $s(I, I') \in \{\pm 1\}$ to the pair (I,I'), by the definition

$$s(I, I') = (-1)^{\#\{j \in I \,/\, i > i_\sigma\}}.$$

It can be readily seen that if $\tau \geq 3$ and σ_1, σ_2 are distinct elements of $\{2, 3, \cdots, \tau\}$, and if $I_1 = I \setminus \{i_{\sigma_1}\}$, $I_2 = I \setminus \{i_{\sigma_2}\}$, $I' = I \setminus \{i_{\sigma_1}, i_{\sigma_2}\}$, then

$$\partial(I_1, I')\partial(I, I_1) = \partial(I_2, I')\partial(I, I_2)$$

and

$$s(I_1, I')s(I, I_1) + s(I_2, I')s(I, I_2) = 0.$$

For each pair (τ,σ) with $t \geq \tau > \sigma \geq 1$ we now define a complex $\mathbf{K} = \mathbf{K}(\tau,\sigma)$:

$$K_i = \begin{cases} \coprod_{\substack{I\subseteq\{t,\,t-1,\,\cdots,\,\tau+1,\,\tau,\,\sigma\} \\ \sigma\in I \\ |I|=i+1}} N_I & \text{if } t>i\geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $i \geq 1$ and N_I and $N_{I'}$ are components of K_i and K_{i-1} resp. then the component of the boundary map ∂_i: $K_i \to K_{i-1}$ between N_I and $N_{I'}$ is $s(I, I')\partial(I, I')$ if $I \supseteq I'$, and zero otherwise. It follows from the properties of $s(I, I')$ and $\partial(I, I')$ that K is indeed a complex. In fact we have:

Proposition

(v) $K=K(\tau,\sigma)$ is exact except in dimension zero.

(vi) $H_0(K) \cong N_\sigma / \sum_{t\geq\tau'\geq\tau} \operatorname{im} h_{\tau',\sigma}$.

Proof

Statement (vi) about H_0 is clear. For (v) we argue by induction on $t-\tau$. If $t = \tau$, K reduces to

$$0 \to N_t \otimes H_{t,\sigma} \to N_\sigma \to 0.$$

By (iv) the middle map is a monomorphism, giving (v).

Now suppose that $t > \tau$. Observe that $K(\tau,\sigma)$ is the mapping cone[18] of the map $h\colon K(\tau+1,\tau) \otimes H_{\tau,\sigma} \to K(\tau+1,\sigma)$ of complexes, where h_i is $(-1)^{i+1}$ times the appropriate component of the boundary ∂_{i+1} of $K(\tau,\sigma)$:

---

[18] Explicitly, if $h\colon X \to Y$ is a map of complexes, the mapping cone of h is the complex whose i^{th} component is $Y_i \amalg X_{i-1}$, and whose i^{th} differential is the map

$$(y,x) \mapsto (\partial_i y + (-1)^i h_{i-1}x,\ \partial_{i-1}x) \qquad x\in X_{i-1},\ y\in Y_i.$$

$$\to \coprod_{t \geq \rho > \rho' \geq \tau+1} N\{\rho,\rho',\sigma\} \to \coprod_{t \geq \rho \geq \tau+1} N\{\rho,\sigma\} \to N_\sigma \to 0$$

$$\uparrow h_2 \qquad \uparrow h_1 \qquad \uparrow h_0$$

$$\to \coprod_{t \geq \rho > \rho' \geq \tau+1} N\{\rho,\rho',\tau,\sigma\} \to \coprod_{t \geq \rho \geq \tau+1} N\{\rho,\tau,\sigma\} \to N\{\tau,\sigma\} \to 0$$

By induction and the long exact sequence of the mapping cone, $H_i(K)$ is zero except possibly when $i \in \{0, 1\}$, and there is an exact sequence:

$$0 \to H_1(K) \to N_\tau / \sum_{t \geq \tau' \geq \tau+1} \operatorname{im} h_{\tau',\tau} \to N_\sigma / \sum_{t \geq \tau' \geq \tau+1} \operatorname{im} h_{\tau',\sigma} \to$$

$$\to N_\sigma / \sum_{t \geq \tau \geq \tau} \operatorname{im} h_{\tau',\sigma} \to 0.$$

Taking the alternating sum of the dimensions we get $\dim H_1(K) = 0$ by (iv). □

Assume notation as in III.2.1. We will use the above proposition to write down an $S(\Omega,\Gamma)$-projective resolution of $S(\Omega,\Gamma)^{\lambda} / M_{\alpha,r}$. If $\lambda_b < q+r$ then $M_{\alpha,r}$ is zero, so assume $\lambda_b \geq q+r$. Put $(a_1, a_2, \cdots, a_t) = (0, q+r, q+r+1, \cdots, \lambda_b)$. For $\tau \in \underline{t}$ put $\lambda(\tau) = \lambda - a_\tau\alpha$ and $N_\tau = S(\Omega,\Gamma)^{\lambda(\tau)}$; for $t \geq \tau > \sigma \geq 1$ let $h_{\tau,\sigma}: N_\tau \to N_\sigma$ be the map given by right multiplication by $\xi_{1(\alpha,a_\tau),1(\alpha,a_\sigma)}$. By III.2.6 the maps $h_{\tau,\sigma}$ satisfy (iii) of III.4.1, so we can construct the complex $K = K(2, 1)$ above.

III.4.2 Theorem (projective resolution of $S(\Omega,\Gamma)^{\lambda} / M_{\alpha,r}$)
$S(\Omega,\Gamma)^{\lambda} / M_{\alpha,r}$ has an $S(\Omega,\Gamma)$-projective resolution

$$0 \to K_{t-1} \to K_{t-2} \to \cdots \to K_1 \to K_0 \to S(\Omega,\Gamma)^{\lambda} / M_{\alpha,r} \to 0,$$

where

$$K_i \cong \coprod_{t \geq \tau \geq 1} \left(S(\Omega,\Gamma)^{\lambda(\tau)}\right)^{\binom{\tau-2}{i-1}} \qquad (t > i \geq 0).$$

The boundary maps $\partial_i: K_i \to K_{i-1}$ $(i \geq 1)$ are as described in III.4.1, while $\partial_0: K_0 \to S(\Omega,\Gamma)^{\lambda} / M_{\alpha,r}$ is the canonical projection. Here $\left(S(\Omega,\Gamma)^{\lambda(\tau)}\right)^{\binom{\tau-2}{i-1}}$ denotes a direct sum of $\binom{\tau-2}{i-1}$ copies of $S(\Omega,\Gamma)^{\lambda(\tau)}$, and we interpret $\binom{-1}{i}$ and $\binom{t}{-1}$ as being 1 if $i = -1$, zero otherwise. Note that this resolution is valid when $\lambda_b < q+r$ if we take $t=1$ and $\lambda(1) = \lambda$.

Proof

If λ is $\{\alpha\}$-dominant it is easy to check that $\lambda-(q+1)\alpha = s_{\alpha}\circ\lambda$ is the highest $\{\alpha\}$-non-dominant weight of the form $\lambda-u\alpha$, i.e. $\lambda-u\alpha$ is $\{\alpha\}$-dominant iff $u \leq q$ [19]. Whether λ is dominant or not, none of the weights $\lambda(2), \lambda(3), \cdots, \lambda(t)$ is $\{\alpha\}$-dominant. From these remarks we see that if $t \geq \sigma > \rho \geq 1$ then $\lambda(\sigma) = \lambda(\rho)-(\,q(\rho) + r(\sigma,\rho)\,)\alpha$, and

$$\sum_{t \geq \sigma} \operatorname{im} h_{\tau,\rho} = M_{\alpha,r(\sigma,\rho)}(\Omega,\Gamma,\lambda(\rho)),$$

where

$$q(\rho) = \begin{cases} 0 & \text{if } \rho > 1 \\ q & \text{if } \rho = 1 \end{cases}$$

and

$$r(\sigma,\rho) = \begin{cases} \sigma - \rho & \text{if } \rho > 1 \\ r+\sigma-2 & \text{if } \rho = 1. \end{cases}$$

Using III.2.8 we deduce that (iv) of III.4.1 holds, so proposition III.4.1 gives the exactness of K in all but dimension zero, and $\operatorname{im} \partial_1 = M_{\alpha,r}$ so we have the required resolution. □

We now use the resolution K to prove a couple of vanishing results for the

[19] It is not true in general that if μ is non-dominant and $\nu \leq \mu$ then ν is non-dominant, e.g. $(2,2,2) \leq (2,4,0)$.

module $S(\Omega,\varnothing)^\lambda / M_{\alpha,r}$ [20]. Our motivation here is that under restricted conditions this module is a Weyl module when r = 1.

III.4.3 Corollary

(i) For any $\{\alpha\}$-dominant weight μ

$$Ext^i_{S(\Omega,\varnothing)} (S(\Omega,\varnothing)^\lambda / M_{\alpha,r} , k(\mu)) = 0 \quad \forall\, i > 0.$$

(ii) $$Tor_i^{S(\Omega,\varnothing)} (S(\Omega,\Gamma) , S(\Omega,\varnothing)^\lambda / M_{\alpha,r}) = \begin{cases} S(\Omega,\Gamma)^\lambda / M_{\alpha,r} & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$ [21]

Proof

Take $\Gamma = \varnothing$ and consider the $S(\Omega,\varnothing)$-projective resolution K of $S(\Omega,\varnothing)^\lambda / M_{\alpha,r}$ as above. If $S(\Omega,\varnothing)^{\lambda(\tau)}$ appears as a component of K_i for $i > 0$, then $\tau > 1$, so as observed in the proof of III.4.2, $\lambda(\tau)$ is not $\{\alpha\}$-dominant. (i) now follows. Suppose $h: S(\Omega,\varnothing)^\mu \to S(\Omega,\varnothing)^\nu$ is given by right multiplication by some element $\xi \in {}^\mu S(\Omega,\varnothing)^\nu$. Application of the functor $S(\Omega,\Gamma) \underset{S(\Omega,\varnothing)}{\otimes}$ produces a map $S(\Omega,\Gamma)^\mu \to S(\Omega,\Gamma)^\nu$ which is also given by right multiplication by ξ. Bearing this in mind it is not hard to check from the construction of K that $S(\Omega,\Gamma) \underset{S(\Omega,\varnothing)}{\otimes} K$ is isomorphic to the corresponding resolution with $(\Omega,\varnothing)$ replaced by (Ω,Γ), giving (ii). □

We have seen that a possible approach to (A') is to study $S(\Omega,\varnothing)$-projective resolutions of the Weyl modules $V(\Omega,\Gamma,\lambda)$. The next proposition shows that this is also pertinent to (B'):

III.4.4 Proposition

Suppose that $\Omega \supseteq \Gamma$ and that for all Ψ-dominant weights λ

$$Tor_i^{S(\Omega,\varnothing)}(S(\Omega,\Gamma), V(\Omega,\Gamma,\lambda)) = 0 \quad \forall\, i > 0.$$

[20] We make the obvious convention that the $M_{\alpha,r}$ appearing in any expression like $S(\Omega,\Gamma')^\lambda / M_{\alpha,r}$ is $M_{\alpha,r}(\Omega,\Gamma',\lambda)$.

[21] The relevance of this result is indicated by III.4.4.

Then

(i) For all $V \in \mathrm{mod}S(\Omega,\Gamma)$, $\mathrm{Tor}_i^{S(\Omega,\emptyset)}(S(\Omega,\Gamma), V) \cong \begin{cases} V & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$

(ii) For all $V,V' \in \mathrm{mod}S(\Omega,\Gamma)$, $\mathrm{Ext}^i_{S(\Omega,\Gamma)}(V, V') \cong \mathrm{Ext}^i_{S(\Omega,\emptyset)}(V, V')$.

Proof

(i) Since $\Omega \supseteq \Gamma$ the case $i = 0$ follows from the fully faithfulness of restriction (see III.3.5), which shows that the natural map $S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes} V \to V$ is an isomorphism for all $V \in \mathrm{mod}S(\Omega,\Gamma)$. By II.3.4 and the homology long exact sequence

$$\mathrm{Tor}_i^{S(\Omega,\emptyset)}(S(\Omega,\Gamma), S(\Omega,\Gamma)) = 0 \quad \forall\, i > 0.$$

The result for arbitrary V now follows by dimension shifting, using the fact that the functor $S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes}$ preserves the exactness of sequences of $S(\Omega,\Gamma)$-modules.

(ii) Using (i) we see that if $V \in \mathrm{mod}S(\Omega,\Gamma)$, the functor $S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes}$ takes any $S(\Omega,\emptyset)$-projective resolution of V to an $S(\Omega,\Gamma)$-projective resolution of V, and this easily implies the result we want (cf. the proof of II.3.16). □

Recall statement (C) from the introduction:

(C) $$\begin{cases} \mathrm{Ext}^i_{S(\Omega,\emptyset)}(V(\Omega,\Gamma,\lambda), k(\mu)) = 0 \quad \forall\, i > 0, \forall\, \Psi\text{-dominant weights } \mu. \\ \mathrm{Tor}_i^{S(\Omega,\emptyset)}(S(\Omega,\Gamma), V(\Omega,\Gamma,\lambda)) = 0 \quad \forall\, i > 0. \end{cases}$$

III.4.5 Corollary

If (C) holds for all Ψ-dominant weights λ then (A') and (B') are true.

Proof

For (A') this follows from II.3.16, while for (B') it is an immediate consequence of the preceding proposition. □

III.4.6 Theorem

Let λ be a Ψ-dominant weight and put

$$\Psi_{\lambda} = \{\alpha\in\Psi \,/\, s_{\alpha}\circ\lambda \text{ is polynomial}\} = \{\alpha_b\in\Psi \,/\, \lambda_b \geq \lambda_{b+1} \geq 1\}.$$

(i) If λ is such that $|\Psi_{\lambda}| \leq 1$ then (C) holds.

(ii) (A') and (B') hold when $|\Psi| \leq 1$. In particular (A) and (B) hold when $n = 2$.

Proof

(i) Take $\alpha = \alpha_b\in\Psi$. Then (see III.2.1) $M_{\alpha} = M_{\alpha}(\Omega,\varnothing,\lambda)$ is non-zero iff $\lambda_{b+1} \geq 1$, i.e. iff $\alpha\in\Psi_{\lambda}$. Thus by III.3.1, $V(\Omega,\Gamma,\lambda)$ is isomorphic to the quotient of $S(\Omega,\varnothing)^{\lambda}$ by the sum over all $\alpha\in\Psi_{\lambda}$ of the submodules M_{α}. If Ψ_{λ} is empty $V(\Omega,\Gamma,\lambda)$ is projective as an $S(\Omega,\varnothing)^{\lambda}$-module, and (C) clearly holds. If $\Psi_{\lambda} = \{\alpha\}$

$$V(\Omega,\Gamma,\lambda) \cong S(\Omega,\varnothing)^{\lambda} / M_{\alpha},$$

so (C) follows from III.4.3.

(ii) If $|\Psi| \leq 1$ then certainly $|\Psi_{\lambda}| \leq 1$ so (C) holds for all Ψ-dominant λ by (i). Thus (A') and (B') hold by III.4.5. □

IV: Special Cases

In this final chapter we prove some more partial results relating to (A') and (B'). We have seen (III.4.5) that to prove (A') and (B') it is enough to show that (C) holds for all Ψ-dominant λ. In §1 we show that (C) holds when λ is a Ψ-hook weight (defined below), by producing explicit projective resolutions. In §2 we explain a connection when char(k) = 0 between the resolutions of §1 and the Bernstein–Gelfand–Gelfand resolution of the simple $sl_n(k)$- module of highest weight λ. We define a complex which if exact would prove (A) and (B) in characteristic zero. We show that for $n \leq 3$ this complex is indeed exact. In §3 we derive a character formula for $V(\Omega,\Gamma,\lambda)$ which is related to the Jacobi–Trudi identity for the Schur function. When $\Omega = \Delta$ this formula shows that the Euler characteristic of the complex of §2 is zero. In §4 we prove (A') (in arbitrary characteristic) when $n = 3$.

§1 Hook Weights

IV.1.1 Definitions

We will call a weight λ a ***Ψ-hook weight*** if its Ψ-components $\lambda(\Xi)$ are hook partitions, i.e. if for each Ψ-block Ξ

$$\lambda(\Xi) = (c, 1, \cdots, 1, 0, \cdots, 0) \quad \text{for some } c \in \mathbb{N}.$$

For the rest of this section λ will be a fixed Ψ-hook weight. This implies that λ is Ψ-dominant. If $\alpha, \beta \in \Delta$ we will write $\alpha < \beta$ if $\alpha = \alpha_a$ and $\beta = \alpha_b$ with $a < b$. For a subset $\Theta \subseteq \Delta$ put

$$w_\Theta = \prod_{\alpha \in \Theta} s_\alpha,$$

the product taken according to the order $\alpha_{n-1}, \alpha_{n-2}, \cdots, \alpha_1$ on Δ, i.e. according to

the reverse of the order $<$.

IV.1.2 Lemma

Recall that $\Psi_\lambda = \{\alpha\in\Psi \mid s_\alpha\circ\lambda \text{ is polynomial}\} = \{\alpha_a\in\Psi \mid \lambda_a \geq \lambda_{a+1} \geq 1\}$.

(i) Take $w\in W_\Psi$. Then $w\circ\lambda$ is polynomial iff $w = w_\Theta$ for some $\Theta\subseteq\Psi_\lambda$.

(ii) If Ξ is a union of Θ-blocks

$$|(w_\Theta\circ\lambda)(\Xi)| = |\lambda(\Xi)|.$$

(iii) If Ξ is a Θ-block,

$$(w_\Theta\circ\lambda)(\Xi) = (0, 0, \cdots, 0, c) \quad \text{for some } c \geq 0, \text{ with } c \geq 1 \text{ if } |\Xi| \geq 2.$$

Proof

Let $\Xi = \{a, a+1, \cdots, b\}$ be a Θ-block. It is easily checked that

$$(w_\Theta\circ\lambda)(\Xi) = \begin{cases} (\lambda_{a+1}-1, \cdots, \lambda_b-1, \lambda_a+b-a) & \text{if } a < b \\ (\lambda_a) & \text{if } a = b. \end{cases}$$

The condition $\Theta\subseteq\Psi_\lambda$ implies that $\lambda_d \geq 1$ whenever $a < d \leq b$, so $w_\Theta\circ\lambda$ is polynomial. Statements (ii) and (iii) follow from the above formula.

Now suppose that $w\in W_\Psi$ and $w\circ\lambda$ is polynomial. We show by induction on $l(w)$ that $w = w_\Theta$ for some $\Theta\subseteq\Psi_\lambda$. If $l(w) = 0$ this is clear, so write $w = s_\alpha w'$ with $\alpha = \alpha_d\in\Psi$, $w'\in W_\Psi$ and $l(w) = l(w')+1$. Put $\mu = w'\circ\lambda$. Then $(w')^{-1}(\alpha)\in\Phi^+\cap\mathbb{Z}\Psi$, so

$$\mu_d - \mu_{d+1} = (w'\circ\lambda,\alpha) = (\lambda,(w')^{-1}\alpha) + (\delta,(w')^{-1}\alpha) - (\delta,\alpha) \geq 0$$

(see I.1.3). This implies that $\mu = w'\circ\lambda$ is also polynomial, since

$$(w\circ\lambda)_a = \begin{cases} \mu_a & \text{if } a\notin\{d, d+1\} \\ \mu_{d+1}-1 & \text{if } a = d \\ \mu_d+1 & \text{if } a = d+1, \end{cases}$$

so by induction $w' = w_\Theta$ for some $\Theta\subseteq\Psi_\lambda$. Let $\Xi = \{a, a+1, \cdots, b\}$ be the Θ-block

containing d. By (iii) we have d = b, and the Ψ-block containing d+1 consists of d+1 alone. Thus $\alpha_d \in \Psi_\lambda \setminus \Theta$, $\alpha_{d+1} \notin \Theta$ and $w = w_{\Theta \cup \{\alpha\}}$. □

IV.1.3 Lemma

For $\Theta \subseteq \Psi_\lambda$ put $\lambda(\Theta) = w_\Theta \circ \lambda$, and let $i(\Theta)$ be the canonical index of weight $\lambda(\Theta)$. Write $\xi_{\Theta,\Theta'}$ for $\xi_{i(\Theta),i(\Theta')}$.

(i) If $\Theta' \subseteq \Theta \subseteq \Psi_\lambda$, then $\xi_{\Theta,\Theta'}$ spans ${}^{\lambda(\Theta)}S(\Psi,\emptyset)^{\lambda(\Theta')}$.

(ii) If $\Theta'' \subseteq \Theta' \subseteq \Theta \subseteq \Psi_\lambda$, then $\xi_{\Theta,\Theta''} = \xi_{\Theta,\Theta'}\xi_{\Theta',\Theta''}$.

Proof

(i) Suppose $\xi_{i,j} \in {}^{\lambda(\Theta)}S(\Psi,\emptyset)^{\lambda(\Theta')}$. We may assume that $i = i(\Theta)$, and that the restriction of j to each $\lambda(\Theta)$-row is canonical. We will show that $j = i(\Theta')$. Let Ξ be a Θ-block. We claim that $i_\varphi \in \Xi$ iff $j_\varphi \in \Xi$. By induction we may suppose that this is true for all Θ-blocks Ξ' which are earlier than Ξ in the natural ordering, i.e. for which $a \in \Xi$, $a' \in \Xi'$ implies that $a > a'$. Suppose $i_\varphi \in \Xi$. By the inductive assumption $j_\varphi \geq \min \Xi$, so since $j_\varphi \leq i_\varphi$ we must have $j_\varphi \in \Xi$. Since $\Theta' \subseteq \Theta$, Ξ is a union of Θ'-blocks, so by IV.1.2(ii)

$$|\{\varphi \in \underline{f} \,/\, i_\varphi \in \Xi\}| = |\lambda(\Xi)| = |\{\varphi \in \underline{f} \,/\, j_\varphi \in \Xi\}|,$$

and the claim is established. Put $a = \max \Xi$. By IV.1.2(iii), $\{\varphi \in \underline{f} \,/\, i_\varphi \in \Xi\}$ is the a^{th} $\lambda(\Theta)$-row, and it follows that $j = i(\Theta')$. It is clear from this discussion that $\xi_{\Theta,\Theta'}$ is indeed itself an element of ${}^{\lambda(\Theta)}S(\Psi,\emptyset)^{\lambda(\Theta')}$.

(ii) By (i), $\xi_{\Theta,\Theta''}\xi_{\Theta',\Theta''}$ is a scalar multiple of $\xi_{\Theta,\Theta''}$, the scalar in question being the number of $s \in I(n,f)$ satisfying

$$(i(\Theta), i(\Theta')) \sim (i(\Theta), s)$$

and

$$(i(\Theta'), i(\Theta'')) \sim (s, i(\Theta'')).$$

The second condition implies that s can be obtained from $i(\Theta')$ by a permutation within the $\lambda(\Theta'')$-rows. The proof of (i) shows that $i(\Theta')$ is constant on these, so the only such s is $i(\Theta')$ itself. □

IV.1.4 Lemma

For $\Theta' \subseteq \Theta \subseteq \Psi_\lambda$ define an $S(\Omega,\Gamma)$-map

$$\psi(\Theta,\Theta'): S(\Omega,\Gamma)^{\lambda(\Theta)} \to S(\Omega,\Gamma)^{\lambda(\Theta')}$$
$$\xi \mapsto \xi \cdot \xi_{\Theta,\Theta'}.$$

(i) If $\Theta'' \subseteq \Theta' \subseteq \Theta \subseteq \Psi_\lambda$ then $\psi(\Theta',\Theta'')\psi(\Theta,\Theta') = \psi(\Theta,\Theta'')$.

(ii) If $\Theta \setminus \Theta' = \{\alpha\}$, with $\alpha > \beta$ for all $\beta \in \Theta'$, then $\psi(\Theta,\Theta')$ is an injection whose image is $M_\alpha(\Omega,\Gamma,\lambda(\Theta'))$.

(iii) Every $\psi(\Theta,\Theta')$ is injective.

Proof

(i) This follows from IV.1.3(ii).

(ii) Let $\alpha = \alpha_b$. The hypothesis of (ii) implies firstly that $w_\Theta = s_\alpha w_{\Theta'}$, so $\lambda_\Theta = s_\alpha \circ \lambda(\Theta')$, and secondly that $\lambda(\Theta')_{b+1} = \lambda_{b+1}$. Since λ is a Ψ-hook weight and $\alpha \in \Psi_\lambda$ we have $\lambda_{b+1} = 1$, and thus $(s_\alpha \circ \lambda(\Theta'))_b = 0$. It follows from the definition of $M_\alpha(\Omega,\Gamma,\lambda(\Theta'))$ that it is generated by $\xi_{\Theta,\Theta'}$ (since in the notation of III.2.1 $\lambda(\Theta')_b = q+1$), and so $\mathrm{im}\psi(\Theta,\Theta')$ is indeed $M_\alpha(\Omega,\Gamma,\lambda(\Theta'))$. $\psi(\Theta,\Theta')$ is injective by III.2.8 which shows that $\dim M_\alpha(\Omega,\Gamma,\lambda(\Theta')) = \dim S(\Omega,\Gamma)^{\lambda(\Theta)}$.

(iii) By (i) it is enough to show that each $\psi(\Theta,\emptyset)$ is injective. By (i) again, any $\psi(\Theta,\emptyset)$ can be written as a composite of maps of the type considered in (ii), so is injective. □

IV.1.5 Projective Resolutions associated with Hook Weights

In this subsection we will construct (for λ a Ψ-hook weight) a projective resolution of $S(\Omega,\Gamma)^\lambda / \sum_{\alpha \in \Psi_\lambda} M_\alpha$. The latter module is a Weyl module under appropriate conditions (see remark (i) following the theorem below). To facilitate the proof of exactness we will define more general complexes.

Suppose that $\Theta' \subseteq \Theta \subseteq \Psi_\lambda$ satisfy

(a) $\qquad \alpha < \beta$ for all $\alpha \in \Theta'$ and all $\beta \in \Theta \setminus \Theta'$.

We define a complex $\mathbf{K} = \mathbf{K}(\Theta,\Theta')$ of $S(\Omega,\Gamma)$-modules which bears a formal

similarity to the complex of III.4.1. Writing S^{Σ} for $S(\Omega,\Gamma)^{\lambda(\Sigma)}$ if $\Sigma\subseteq\Psi_\lambda$, put

$$K_i = \coprod_{\substack{\Theta'\subseteq\Sigma\subseteq\Theta \\ |\Sigma\setminus\Theta'| = i}} S^{\Sigma}.$$

If $i \geq 1$ and S^{Σ} and $S^{\Sigma'}$ are components of K_i and K_{i-1} resp., the component of the boundary map ∂_i: $K_i \to K_{i-1}$ between them is $s(\Sigma,\Sigma')\psi(\Sigma,\Sigma')$ if $\Sigma\supseteq\Sigma'$, zero otherwise. Here

$$s(\Sigma,\Sigma') = (-1)^{\#\{\beta\in\Sigma\ /\ \beta > \alpha\}}, \text{ where } \{\alpha\} = \Sigma\setminus\Sigma'.$$

Using IV.1.4(i) it is easy to see that this defines a complex.

Theorem

(i) $H_0(K) = \dfrac{S(\Omega,\Gamma)^{\lambda(\Theta')}}{\sum_{\gamma\in\Theta\setminus\Theta'} M_\gamma}$.

(ii) K is exact in all non-zero dimensions.

Proof

(i) This follows from IV.1.4(ii).

(ii) This is similar to the corresponding proof in III.4.1: we argue by induction on $t = |\Theta\setminus\Theta'|$ using a mapping cone construction. Let $n(\Theta,\Theta')$ be the number of $\lambda(\Theta')$-tableaux ξ with values in $\underline{n}$ which are row semi-standard, $(\Theta\setminus\Theta')$-column standard and which satisfy $(\xi(a,b),a)\in[\Omega|\Gamma]\ \ \forall(a,b)\in[\lambda(\Theta')]$ [22]. We include among our inductive hypotheses the statement

(b) $$\dim H_0K = n(\Theta,\Theta')$$ [23]

For $t = 0$, K is the sequence $0 \to S(\Omega,\Gamma)^{\lambda(\Theta)} \to 0$, so (ii) and (b) are certainly true. Now suppose that $t \geq 1$. Let α be the least element of $\Theta\setminus\Theta'$ with respect to the order $<$ on Δ. A routine verification shows that $K(\Theta,\Theta')$ is the mapping cone of the map of

[22] I.e. in the notation of IV.3.3 ξ is adapted to $[\Omega|\Gamma]$.

[23] This is a special case of a general result - see remark (i) following III.2.5.

complexes

$$h: K(\Theta, \Theta'\cup\{\alpha\}) \to K(\Theta\backslash\{\alpha\}, \Theta'),$$

where h_i is $(-1)^{i+1}$ times the appropriate component of the boundary ∂_{i+1} of $K(\Theta,\Theta')$:

$$\begin{array}{ccccccccc}
\to & \coprod_{\substack{\Theta'\subseteq\Sigma\subseteq\Theta\backslash\{\alpha\} \\ |\Sigma\backslash\Theta'|=2}} S^{\Sigma} & \to & \coprod_{\substack{\Theta'\subseteq\Sigma\subseteq\Theta\backslash\{\alpha\} \\ |\Sigma\backslash\Theta'|=1}} S^{\Sigma} & \to & S^{\Theta'} & \to 0 \\
& \uparrow h_2 & & \uparrow h_1 & & \uparrow h_0 & \\
\to & \coprod_{\substack{\Theta'\subseteq\Sigma\subseteq\Theta\backslash\{\alpha\} \\ |\Sigma\backslash\Theta'|=2}} S^{\Sigma\cup\{\alpha\}} & \to & \coprod_{\substack{\Theta'\subseteq\Sigma\subseteq\Theta\backslash\{\alpha\} \\ |\Sigma\backslash\Theta'|=1}} S^{\Sigma\cup\{\alpha\}} & \to & S^{\Theta'\cup\{\alpha\}} & \to 0
\end{array}$$

Each of the pairs of sets $(\Theta, \Theta'\cup\{\alpha\})$ and $(\Theta\backslash\{\alpha\}, \Theta')$ satisfies condition (a), and the respective set differences both have size $t-1$. By induction and the long exact sequence of the mapping cone, K is exact except possibly in dimensions 0 and 1, and there is an exact sequence

(c) $$0 \to H_1K \to H_0K(\Theta, \Theta'\cup\{\alpha\}) \to H_0K(\Theta\backslash\{\alpha\}, \Theta') \to H_0K \to 0.$$

$\lambda(\Theta')$ is $(\Theta\backslash\Theta')$-dominant, so by I.1.6, $n(\Theta,\Theta')$ is the number of $(\Theta\backslash\Theta')$-standard basis elements $\xi_{i,l} \in S(\Omega,\Gamma)^{\lambda(\Theta')}$. Thus by remark (i) following III.2.5, $\dim H_0K \le n(\Theta, \Theta')$. Taking the alternating sum of the dimensions in (c) and using induction we have

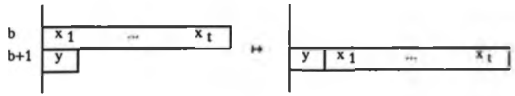
$$n(\Theta,\Theta') \ge \dim H_0K = n(\Theta\backslash\{\alpha\}, \Theta') - n(\Theta, \Theta'\cup\{\alpha\}) + \dim H_1K.$$

It is therefore enough to show that

(d) $$n(\Theta,\Theta') + n(\Theta, \Theta'\cup\{\alpha\}) = n(\Theta\backslash\{\alpha\}, \Theta'),$$

which we do using a combinatoric argument.

Let $\alpha = \alpha_b$. Condition (a) implies that $\lambda(\Theta')_b \geq 1$, and $\lambda(\Theta')_{b+1} = 1$. The shape $[\lambda(\Theta'\cup\{\alpha\})]$ is thus obtained from $[\lambda(\Theta')]$ by removing row b and adding it onto row b+1. Define a map from $\lambda(\Theta')$-tableaux to $\lambda(\Theta'\cup\{\alpha\})$-tableaux by taking row b of a $\lambda(\Theta')$-tableau and putting it onto the end of row b+1 to obtain a $\lambda(\Theta'\cup\{\alpha\})$-tableau (cf. the map used in the proof of III.2.8):



This map induces a bijection from the set of $\lambda(\Theta')$-tableaux which are row semi-standard, $\Theta\setminus(\Theta'\cup\{\alpha\})$-column standard but not $\{\alpha\}$-column standard to the set of $\lambda(\Theta'\cup\{\alpha\})$-tableaux which are row semi-standard and $\Theta\setminus(\Theta'\cup\{\alpha\})$-column standard [24], which establishes (d). □

Remarks

(i) $K(\Theta,\Theta')$ is a deleted $S(\Omega,\Gamma)$-projective resolution of the module

$$S(\Omega,\Gamma)^{\lambda(\Theta')} / \sum_{\alpha\in\Theta\setminus\Theta'} M_\alpha.$$

Take a subset $\Gamma'\subseteq\Psi$ and consider the resolution $K(\Psi_\lambda,\emptyset)$ with (Ω,Γ') taking the place of (Ω,Γ). $M_\alpha(\Omega,\Gamma',\lambda)$ is zero if $\alpha\in\Psi\setminus\Psi_\lambda$, so by III.3.1 $K(\Psi_\lambda,\emptyset)$ is a deleted $S(\Omega,\Gamma')$-projective resolution of

$$S(\Omega,\Gamma)^\lambda / \sum_{\alpha\in\Psi_\lambda} M_\alpha \cong V(\Omega,\Gamma,\lambda).$$

For the case $\Omega = \Gamma = \Gamma' = \Delta$, essentially the same resolution of $V(\Delta,\Delta,\lambda)$ is given in [Ma]. The proof of exactness is similar in spirit to the one above, involving the

[24] Note that if $\mu\in\Lambda(n,f)$, $\Sigma\subseteq\Delta$, and μ is Σ-dominant, then a μ-tableau is Σ-column standard iff it is $\{\beta\}$-column standard for all $\beta\in\Sigma$.

splicing together of simpler complexes using mapping cones, although the 'component complexes' in [Ma] are not in general the same ones that appear in the proof above.

(ii) Since each of the maps $\psi(\Sigma,\Sigma')$ is injective (IV.1.4(iii)), we could replace S^{Σ} in the definition of $K(\Theta,\Theta')$ by its image in $S(\Omega,\Gamma)^{\lambda(\Theta')}$ and each component of the boundary by $\pm$ an inclusion map.

(iii) When $\Gamma = \emptyset$, $K = K(\Theta,\Theta')$ is in fact a minimal projective resolution. This can be seen as follows: let $hd: \mathrm{mod}S(\Omega,\emptyset) \to \mathrm{mod}S(\Omega,\emptyset)$ be the head functor $V \mapsto V/\mathrm{rad}V$. Using K to compute $(L_i hd)(H_0 K)$ we find that

$$(L_i hd)(H_0 K) \cong \coprod_{\substack{\Theta' \subseteq \Sigma \subseteq \Theta \\ |\Sigma \setminus \Theta'| = i}} k(\lambda(\Sigma)),$$

(since $\lambda(\Sigma) = \lambda(\Sigma')$ iff $\Sigma = \Sigma'$), i.e. $(L_i hd)(H_0 K) \cong hd(K_i)$. Thus K has no redundant summand.

IV.1.6 Corollary

(C) holds when λ is a Ψ-hook weight, i.e.

(i) $\mathrm{Ext}^i_{S(\Omega,\emptyset)}(\, V(\Omega,\Gamma,\lambda),\, k(\mu)\,) = 0 \quad \forall\, \Psi$-dominant weights μ, $\forall\, i > 0$.

(ii) $\mathrm{Tor}_i^{S(\Omega,\emptyset)}(S(\Omega,\Gamma),\, V(\Omega,\Gamma,\lambda)) = 0 \quad \forall\, i > 0$.

Proof

(i) Take $\Theta = \Psi_\lambda$ and $\Theta' = \emptyset$, and consider the complex $K = K(\Theta,\Theta')$ with $(\Omega,\emptyset)$ in place of (Ω,Γ). As in remark (i) above K is an $S(\Omega,\emptyset)$-projective resolution of $V(\Omega,\Gamma,\lambda)$. If $\emptyset \neq \Sigma \subseteq \Psi_\lambda$ the weight $\lambda(\Sigma)$ is not Ψ-dominant by IV.1.2(iii) so for $i > 0$ the head of K_i contains no Ψ-dominant weights. The result follows.

(ii) Take K as in (i). It is clear that $S(\Omega,\Gamma) \underset{S(\Omega,\emptyset)}{\otimes} K$ is isomorphic to the corresponding complex with (Ω,Γ) in place of $(\Omega,\emptyset)$, so its homology is zero in all non-zero dimensions. □

§2 Connection with the Bernstein-Gelfand-Gelfand Resolution

In this section we assume that char(k) = 0. We will show that when $\Omega = \Delta$ the resolution of $V(\Delta,\Delta,\lambda)$ of IV.1.5 is a special case of a complex which exists for all dominant weights λ, not just for hook partitions. This complex is obtained from the Bernstein-Gelfand-Gelfand resolution of the simple $sl_n(k)$-module of highest weight λ. Unfortunately we do not know whether this complex is exact in general. Although for hook partitions the resolution of IV.1.5 is characteristic-free, there is certainly no analogous resolution in prime characteristic for general dominant weights; this can be seen by considering the case $n = 2$.

IV.2.1 Verma Modules

Let $\mathfrak{g} = sl_n(k)$ be the simple Lie algebra of $n \times n$ matrices over k of trace zero. We recall some results on Verma modules (see e.g. [Di; Ch.7]). We will not state these results in their greatest generality. Let $\mathfrak{h}$ be the subalgebra of diagonal matrices in $\mathfrak{g}$, a splitting Cartan subalgebra. There is an obvious map $\Lambda(n) \to \mathfrak{h}^* = \mathrm{Hom}_k(\mathfrak{h}, k)$, and we will write the image of $\lambda \in \Lambda(n)$ by the same symbol:

$$\lambda(h) = \lambda_1 h_1 + \cdots + \lambda_n h_n \qquad \lambda \in \Lambda(n),\ h = \mathrm{diag}(h_1, \cdots, h_n) \in \mathfrak{h}$$

In this way Φ is identified with the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and the image of $\Lambda(n)$ is the set of weights of this root system.

Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition of $\mathfrak{g}$ associated with Δ, i.e.

$$\mathfrak{n}^- = \sum_{\alpha \in \Phi^-} \mathfrak{g}^\alpha, \qquad \mathfrak{n}^+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}^\alpha.$$

Put $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$, and $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$. $\mathfrak{n}^-$, $\mathfrak{b}^-$, $\mathfrak{n}^+$ and $\mathfrak{b}^+$ are the subalgebras consisting of all matrices in $\mathfrak{g}$ which are respectively strictly lower triangular, lower triangular, strictly upper triangular and upper triangular.

For $\lambda \in \mathfrak{h}^*$ let $k(\lambda)$ denote the one-dimensional $\mathfrak{h}$-module of weight λ. This should not cause confusion with our previous use of the notation $k(\lambda)$. We view $k(\lambda)$

as a b^+-module via the canonical surjection $b^+ \to h$. $k(\lambda)$ can be similarly regarded as a b^--module. If a is a Lie algebra we will denote its universal enveloping algebra by $U(a)$. Put

$$M(\lambda) = U(g) \underset{U(b^+)}{\otimes} k(\lambda),$$

the ***Verma module*** of highest weight λ [25]. The map

(a)
$$\begin{array}{c} U(n^-) \to M(\lambda) \\ u \mapsto u \otimes 1 \end{array}$$

is an isomorphism of $U(n^-)$-modules.

Suppose $\lambda \in \Lambda(n)$ is dominant. For each $w \in W$, $\mathrm{Hom}_g(M(w\circ\lambda), M(\lambda))$ is one-dimensional, and every non-zero element therein is an injection [Di; 7.6.6, 7.6.8], so we can (and will) consider $M(w\circ\lambda)$ to be a submodule of $M(\lambda)$.

For $w,w' \in W$ we will write $w \le w'$ if each reduced expression for w contains as a subsequence a reduced expression for w'. This is a partial order, the reverse of the Bruhat partial order on W. If $w \le w'$ then $M(w\circ\lambda) \le M(w'\circ\lambda)$ as submodules of $M(\lambda)$ [Di; 7.7.7].

We refer to [BGG; 10.3, 10.4] for the following result:

IV.2.2 Lemma

(i) Let $w_1, w_2 \in W$ with $l(w_1) = l(w_2)+2$. Then the number of elements $w \in W$ with $w_1 < w < w_2$ is either zero or two.

(ii) It is possible to attach a sign $s(w,w') = \pm 1$ to each pair $w,w' \in W$ with $w \le w'$ and $l(w)=l(w')+1$ in such a way that whenever

$$w_1, w_2 \in W, \quad l(w_1)=l(w_2)+2, \quad w_1 < w < w_2, \; w_1 < w' < w_2, \quad \text{and } w \ne w'$$

we have

$$s(w_1,w)s(w,w_2) + s(w_1,w')s(w',w_2) = 0. \; \square$$

[25] In [Di] this module is denoted $M(\lambda+\delta)$.

IV.2.3 The Bernstein-Gelfand-Gelfand Resolution

Suppose that $\lambda \in \Lambda(n)$ is dominant. Define a complex $\mathbf{C(\lambda)}$ by

$$C(\lambda)_i = \coprod_{\substack{w \in W \\ l(w)=i}} M(w \circ \lambda).$$

If $i \geq 1$ and $w, w' \in W$ with $l(w) = i$, $l(w') = i-1$, the component of the boundary map ∂_i between $M(w \circ \lambda)$ and $M(w' \circ \lambda)$ is $s(w,w')$ times the inclusion map if $w \leq w'$, zero otherwise. IV.2.2 shows that this does indeed define a complex. In fact

$$H_i C(\lambda) \cong \begin{cases} L(\lambda) & \text{if } i = 0 \\ 0 & \text{if } i > 0, \end{cases}$$

where $L(\lambda) = M(\lambda)/\mathrm{rad}M(\lambda)$ is the simple $\mathfrak{g}$-module of highest weight λ. Thus $C(\lambda)$ is a deleted projective resolution of $L(\lambda)$. This is proved in [BGG] for the case $k = \mathbb{C}$. That it holds for general (characteristic zero) k can be seen by noting that the complex $C(\lambda)$ and the module $L(\lambda)$ can be obtained by base change from the corresponding complex and module over $\mathbb{Q}$, so that the result for any one particular k implies the result for all k.

IV.2.4 Definition of the Complex $K(P,\lambda)$

We can identify $\mathfrak{b}^-$ in the usual way with a Lie subalgebra of the dual algebra $k[B^-]^*$ of the coordinate ring of B^-. Composing with the canonical epimorphism $k[B^-]^* \to S(B^-)$ and using the universal property of $U(\mathfrak{b}^-)$ we get a k-algebra map

$$\theta: U(\mathfrak{b}^-) \to S(B^-),$$

which is in fact an epimorphism. The image of $U(\mathfrak{h})$ under θ is $S(T)$, and if $V \in \mathrm{mod}S(T)$ is considered as a $U(\mathfrak{h})$-module via θ, the weight spaces of V as defined for $S(T)$ coincide with those as defined for $U(\mathfrak{h})$. θ defines a restriction functor from $S(B^-)$-modules to $U(\mathfrak{b}^-)$-modules whose left adjoint is the functor $S(B^-) \underset{U(\mathfrak{b}^-)}{\otimes}$.

Take a weight $\mu \in \Lambda(n,f)$. The composite map

$$U(\mathfrak{g}) \underset{U(\mathfrak{b}^+)}{\otimes} k(\mu) \to U(\mathfrak{n}^-) \to U(\mathfrak{b}^-) \underset{U(\mathfrak{h})}{\otimes} k(\mu),$$

given by $u \otimes 1 \mapsto u \mapsto u \otimes 1$ if $u \in U(\mathfrak{n}^-)$ is an isomorphism of $U(\mathfrak{b}^-)$-modules, so

$$S(B^-) \underset{U(\mathfrak{b}^-)}{\otimes} M(\mu) \cong S(B^-)^{\mu}$$

as $S(B^-)$-modules. Suppose $\lambda \in \Lambda(n,f)$ is dominant and P is any parabolic subgroup of G containing B^-. Form the complex of $S(P)$-modules

$$K(P,\lambda) = S(P) \underset{U(\mathfrak{b}^-)}{\otimes} C(\lambda).$$

IV.2.5 Proposition

Write $P = P_{\Delta,\Gamma}$ with $\Gamma \subseteq \Delta$. Then

(i) $K(P,\lambda)_l \cong \coprod_{\substack{w \in W \\ l(w)=i}} S(P)^{w\circ\lambda}$.

(ii) Suppose $w \in W$, $\alpha \in \Delta$ and $l(s_\alpha w) = l(w) + 1$. Then the map $S(P)^{(s_\alpha w\circ\lambda)} \to S(P)^{w\circ\lambda}$ corresponding to the inclusion $M(s_\alpha w\circ\lambda) \leq M(w\circ\lambda)$ is an injection whose image is $M_\alpha(\Delta,\Gamma,w\circ\lambda)$. (cf. IV.1.4(ii).)

(iii) Whenever $w,w' \in W$ with $w \leq w'$ the map $S(P)^{w\circ\lambda} \to S(P)^{w'\circ\lambda}$ corresponding to the inclusion $M(w\circ\lambda) \leq M(w'\circ\lambda)$ is an injection. (cf. IV.1.4(iii).)

(iv) $H_0K(P,\lambda) \cong V(\Delta,\Delta,\lambda)$.

Proof

(i) This follows once we observe that for $\mu \in \Lambda(n,f)$

$$S(P) \underset{U(\mathfrak{b}^-)}{\otimes} M(\mu) \cong S(P) \underset{S(B^-)}{\otimes} S(B^-) \underset{U(\mathfrak{b}^-)}{\otimes} M(\mu) \cong S(P)^{\mu}.$$

(ii) Suppose $\mu \in \Lambda(n,f)$, $\alpha = \alpha_b \in \Delta$ and that $t = (\mu+\delta,\alpha) \geq 0$. Let $X_{-\alpha}$ denote the element of $\mathfrak{n}^-$ which has a one in the (b+1,b)-position and zeroes elsewhere. The endomorphism of $U(\mathfrak{n}^-)$ given by right multiplication by $X_{-\alpha}^t$ corresponds to an

injection $M(s_\alpha \circ \mu) \to M(\mu)$ once identifications are made as in IV.2.1(a) (see [Di; 7.1.15]). It follows that the corresponding map $S(P)^{s_\alpha \circ \mu} \to S(P)^{\mu}$ is given by right multiplication by the $(s_\alpha \circ \mu, \mu)$-weight component of $\theta(X^t_{-\alpha})$, i.e. the component of weight $(s_\alpha \circ \mu, \mu)$ when $\theta(X^t_{-\alpha})$ is written as a sum of two-sided weight vectors. Let l be the canonical index of weight μ. It is a straightforward exercise[26] to show that the $(s_\alpha \circ \mu, \mu)$-weight component of $\theta(X^t_{-\alpha})$ is $t^{l \cdot \xi_{l(\alpha,t),l}}$ (notation as in III.2.1).

The condition $l(s_\alpha w) = l(w)+1$ implies that

$$(w \circ \lambda + \delta, \alpha) = (\lambda, w^{-1}\alpha) + (\delta, w^{-1}\alpha) \geq 1$$

(see I.1.3). We can thus apply the foregoing discussion with $\mu = w \circ \lambda$ to conclude that the map $S(P)^{w \circ \lambda} \to S(P)^{\lambda}$ is given by multiplication on the right by some non-zero scalar multiple of $\xi_{l(\alpha,t),l}$; the required result follows by III.2.8.

(iii) It is enough to prove this when $w' = 1$, where the result follows from (ii) by considering a reduced expression for w.

(iv) $H_0K(P,\lambda)$ is the quotient of $S(P)^{\lambda}$ by the sum of the images of the maps $S(P)^{s_\alpha \circ \lambda} \to S(P)^{\lambda}$ as α ranges over Δ. By (ii) these images are the various $M_\alpha(\Delta,\Gamma,\lambda)$, so the result follows from III.3.1. □

Remarks

(i) The proposition shows that we can replace each component $S(P)^{w \circ \lambda}$ of $K(P,\lambda)$ by its image in $S(P)^{\lambda}$, and the components of the boundary maps by ± inclusions. It is tempting to try to define the complex $K(P,\lambda)$ without mention of Lie algebras: for $1 \neq w \in W$ take a reduced expression

$$w = {}^{s}\alpha_1 \circ {}^{s}\alpha_{l-1} \circ \cdots \circ {}^{s}\alpha_1 \quad (\alpha_1, \cdots, \alpha_l \in \Delta).$$

---

[26] Using the fact that $\mathfrak{b}^-$ acts by 'ε-point derivations' on $k[B^-]$, i.e. if $X \in \mathfrak{b}^-$

$$X(cc') = X(c)\varepsilon(c') + \varepsilon(c)X(c') \quad \forall c, c' \in k[B^-],$$

one shows that $\theta(X_{-\alpha})$ is the sum of all $\xi_{l(\alpha,t),l}$, where l ranges over all canonical indices in $l(n,f)$ which have at least one value equal to b. Now use III.2.6.

Put $\lambda(0) = \lambda$, and $\lambda(i) = s_{\alpha_i} \circ \lambda(i-1)$ for $i \in I$. Let $\xi(i)$ be the unique standard basis element in $\lambda(i)S(B)^{\lambda(i-1)}$ and put $M_w = S(P)\xi(l)\cdots\xi(1)$, a submodule of $S(P)^{\lambda}$. We then define

$$K(P,\lambda)_i = \coprod_{\substack{w \in W \\ l(w)=i}} M_w,$$

letting the components of the boundary maps be $s(w,w')$ times the appropriate inclusions. Although this prescription works in principle, we know of no way without recourse to Lie algebras of showing the two crucial facts:

(a) M_w is independent of the choice of reduced expression.

(b) $M_w \leq M_{w'}$ if $w \leq w'$.

(ii) If we follow the procedure of (i) when λ is a hook partition, where we can handle (a) and (b) directly – see IV.1.2(i) and IV.1.3, we get the complex $K(\Delta,\varnothing)$ of IV.1.5. More precisely we get the alternative version of $K(\Delta,\varnothing)$ given in remark (ii) of IV.1.5. We may have to adopt a new sign convention in IV.1.5. Thus for hook partitions at least, $H_iK(P,\lambda) = 0$ if $i > 0$.

The complex $K(G,\lambda)$ is essentially the same as the complex of [Z; Example 1]. A related complex is constructed in [Ak]. In each of these papers the complex in question is shown to be exact in positive dimensions. We outline a proof of this fact for $K(G,\lambda)$, partly to indicate why the proof does not readily generalize to arbitrary P.

IV.2.6 Theorem

$H_iK(G,\lambda) = 0 \ \forall\, i > 0$.

Proof (sketch)

If V and V′ are finite-dimensional right and left $\mathfrak{g}$-modules resp. then the groups $H_i(n^-,V)$ and $\mathrm{Tor}_i^{U(n^-)}(V, V')$ carry natural left $\mathfrak{h}$-module structures. If $\mathfrak{a}$ is a Lie algebra, denote by τ the principal anti-automorphism of $U(\mathfrak{a})$, induced by negation in

a. τ induces an isomorphism $V \mapsto V^\tau$ between the categories of left and right $U(a)$-modules. We can use the resolution $C(\lambda)^\tau$ of $L(\lambda)^\tau$ to deduce a strong version of Bott's theorem (see [Bt; §15], [BGG; 9.11, 10.1], [Ak; §2]):

$$H_i(n^-, L(\lambda)^\tau) \cong \coprod_{\substack{w \in W \\ l(w) = i}} k(w\circ\lambda) \quad \text{as left } h\text{-modules.}$$

Let V be a finite-dimensional right g-module. Since V is semisimple, and since if λ is dominant and $1 \neq w \in W$ then $w\circ\lambda$ is not dominant,

$$^\mu H_i(n^-, V) = 0 \; \forall \; i > 0 \, , \; \forall \text{ dominant weights } \mu.$$

The homology group $H_iK(G, \lambda)$ is isomorphic to the zero weight space of $\mathrm{Tor}_i^{U(n^-)}(S(G), L(\lambda))$ and the latter is isomorphic as an h-module to $H_i(n^-, S(G)\otimes L(\lambda)^\tau)$. Since $S(G)\otimes L(\lambda)^\tau$ is a finite-dimensional right g-module, the zero weight space of $H_i(n^-, S(G)\otimes L(\lambda)^\tau)$ is zero for all $i > 0$. □

We speculate that the complex $K(P,\lambda)$ is exact in all positive degrees. As we have seen, this is the case for hook partitions, and for arbitrary λ when $P = G$. If this were the case generally we could prove (A) and (B) in characteristic zero, by using a similar argument to that of IV.1.6 to show that (C) holds for all dominant weights λ. If $n \leq 3$ all is well:

IV.2.7 Theorem

If $n \leq 3$, $H_iK(P,\lambda) = 0$ for all $i > 0$, so $K(P,\lambda)$ is a deleted $S(P)$-projective resolution of $V(\Delta,\Delta,\lambda)$.

Proof

This is a rather piecemeal argument. For $P = G$ the result is covered by IV.2.6, so suppose $P \neq G$. For $n = 1$ the result is vacuous; for $n = 2$ it follows by IV.2.5(iii). Suppose $n = 3$. The complex $K = K(P,\lambda)$ has the form:

$$0 \to S(P)^{(13)\circ\lambda} \to S(P)^{(123)\circ\lambda} \coprod S(P)^{(132)\circ\lambda} \to$$

$$\to S(P)^{(12)\circ\lambda} \coprod S(P)^{(23)\circ\lambda} \to S(P)^{\lambda} \to 0$$

Firstly consider the case $P = B^-$. By IV.2.5(iii) $H_3K = 0$. Suppose $H_2K \neq 0$ and let μ be a weight of H_2K. Then by IV.2.5(ii), μ is a weight of $S(B^-)^{(23)(12)\circ\lambda} / M_{\alpha_1}$, and this module is isomorphic to $V(\Delta,\{\alpha_1\},(23)(12)\circ\lambda)$ by III.3.1. Thus $(12)\mu \trianglelefteq (23)(12)\circ\lambda$ (since the set of weights of $V(\Delta,\{\alpha_1\},(23)(12)\circ\lambda)$ is closed under the permutation (12)), and in particular $\mu_2 \leq \lambda_2 - 1$. Similarly μ is a weight of $V(\Delta,\{\alpha_2\},(12)(23)\circ\lambda)$ so $(23)\mu \trianglelefteq (12)(23)\circ\lambda$ and $\mu_2 \geq \lambda_2 + 1$, a contradiction. Thus $H_2K = 0$. The proof that $H_1K = 0$ is similar [27].

It remains to treat the case where P is a minimal parabolic subgroup of G containing B^-. Then

$$K(P,\lambda) \cong S(P) \underset{S(B^-)}{\otimes} K(B^-,\lambda),$$

so, since we have established already that $K(B^-,\lambda)$ is a deleted $S(B^-)$-projective resolution of $V(\Delta,\Delta,\lambda)$, $H_iK(P,\lambda) \cong \mathrm{Tor}_i^{S(B^-)}(S(P),V(\Delta,\Delta,\lambda))$ which is zero if $i > 0$ by III.4.6 and III.4.4. □

IV.2.8 Corollary

(A) and (B) hold when $n \leq 3$ and $\mathrm{char}(k) = 0$.

Proof

This just requires a simple modification of the proof of IV.1.6 to show that (C) holds for all dominant weights λ. The above theorem shows that $K(B^-,\lambda)$ is a deleted $S(B^-)$-projective resolution with the properties that the head of $K(B^-,\lambda)_i$ has no dominant weights if $i > 0$, and that $S(G) \underset{S(B^-)}{\otimes} K(B^-,\lambda) \cong K(G,\lambda)$ is exact in positive dimensions. □

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[27] Alternatively we could deduce that $H_1K = 0$ from the fact (to be proved in §3) that the Euler characteristic of K is equal to the character of H_0K.

§3 A Character Formula

Let $\mathbb{Z}\Lambda(n,f)$ denote the free $\mathbb{Z}$-module with basis $\Lambda(n,f)$. If $V \in \mathrm{mod}S(\emptyset,\emptyset)$ we define the ***(formal) character*** of V to be the element $[V]$ of $\mathbb{Z}\Lambda(n,f)$ whose μ-coefficient for $\mu \in \Lambda(n,f)$ is $\dim {}^{\mu}V$. V is determined up to isomorphism as an $S(\emptyset,\emptyset)$-module by its character. If we identify $\mu \in \Lambda(n,f)$ with the monomial

$$X^{\mu} = X_1^{\mu_1} X_2^{\mu_2} \cdots X_n^{\mu_n},$$

$\mathbb{Z}\Lambda(n,f)$ becomes a $\mathbb{Z}$-submodule of the polynomial ring $\mathbb{Z}[X_1, \cdots, X_n]$. For each dominant weight λ, the character of the classical Weyl module $V(\Delta,\Delta,\lambda)$ is the Schur function s_λ, a certain symmetric polynomial. One of the well-known identities involving the Schur functions is the Jacobi-Trudi identity, which expresses s_λ as an integral combination of products of complete symmetric functions (see [Md; I.3]). This identity can be written in the following form:

(a) $$s_\lambda = \sum_{w \in W} \mathrm{sgn}(w) \cdot h_{w\circ\lambda}.$$

Here $h_\mu = h_{\mu_1} \cdot h_{\mu_2} \cdots h_{\mu_n}$, where for $r \in \mathbb{N}_0$, h_r is the r^{th} complete symmetric function, i.e. the sum of all monomials of total degree r. It is easily seen that h_μ is the character of the module $S(\Delta,\Delta)^{\mu}$.

The above formula can be realized by equating to zero the Euler characteristic of the resolution of $V(\Delta,\Delta,\lambda)$ given by the complex $K(G,\lambda)$ of IV.2.4 (cf. [Ak], [Z]).

We will derive a formula analogous to (a) for the character of the Weyl module $V(\Omega,\Gamma,\lambda)$: for each subset $\Gamma' \subseteq \Psi$ we have

(b) $$[V(\Omega,\Gamma,\lambda)] = \sum_{w \in W_\Psi} \mathrm{sgn}(w) \cdot [S(\Omega,\Gamma')^{w\circ\lambda}]$$

We remark that even in the classical case $\Omega = \Gamma = \Delta$, formula (b) gives new information. We can recover the Jacobi-Trudi identity from (b) by taking $\Omega = \Gamma = \Gamma' = \Delta$.

We recall some of the basic properties of distinguished coset representatives for parabolic subgroups of W (see e.g. [Bb; Ex3, p.37]).

IV.3.1 Lemma

Let $\Theta_1 \subseteq \Theta_2 \subseteq \Delta$.

(i) For each $w \in W_{\Theta_2}$ the coset $W_{\Theta_1} \cdot w$ has a unique element of minimal length. We will write $\{\Theta_1|\Theta_2\}$ for the set of these distinguished coset representatives.

(ii) Let $w \in W_{\Theta_2}$. Then

$$w \in \{\Theta_1|\Theta_2\} \text{ iff } w^{-1}(\Theta_1) \subseteq \Phi^+.$$

(iii) If $\Theta_1 \subseteq \Theta_2 \subseteq \Theta_3 \subseteq \Delta$

$$\{\Theta_1|\Theta_3\} = \{\Theta_1|\Theta_2\} \cdot \{\Theta_2|\Theta_3\}. \square$$

IV.3.2 Lemma

Suppose that $\Theta_1 \subseteq \Theta_2 \subseteq \Delta$, and that $\lambda \in \Lambda(n,f)$ is Θ_2-dominant. Then $w \circ \lambda$ is Θ_1-dominant for all $w \in \{\Theta_1|\Theta_2\}$.

Proof

Take $\alpha \in \Theta_1$. We must show that $(w \circ \lambda, \alpha) \geq 0$. We have

$$(w \circ \lambda, \alpha) = (\lambda, w^{-1}\alpha) + (\delta, w^{-1}\alpha) - (\delta, \alpha).$$

By IV.3.1(ii) $w^{-1}\alpha \in \Phi^+$, so $(\delta, w^{-1}\alpha) \geq 1$, whilst $(\delta, \alpha) = 1$. $w^{-1}\alpha$ is a non-negative integer combination of roots in Θ_2, so the Θ_2-dominance of λ implies that $(\lambda, w^{-1}\alpha) \geq 0$. $\square$

IV.3.3 Definitions

As usual fix $\Omega, \Gamma \subseteq \Delta$. For $\Theta \subseteq \Omega$ and $\lambda \in \Lambda(n,f)$ a Θ-dominant weight, define an element $\nu(\Theta,\lambda) = \nu(\Omega,\Gamma,\Theta,\lambda)$ of $\mathbb{Z}\Lambda(n,f)$ by setting its μ-coefficient equal to the number of λ-tableaux $\xi: [\lambda] \to \underline{n}$ satisfying the following conditions:

(i) ξ is row semi-standard.

(ii) ξ is Θ-column standard.

(iii) ξ has *weight* μ, i.e. $\forall \nu \in \underline{n}\ \ \mu_\nu = \#\{(a,b) \in [\lambda] \ / \ \xi(a,b) = \nu\}$.

(iv) ξ is *adapted to* $[\Omega|\Gamma]$ in the sense that $(\xi(a,b),a) \in [\Omega|\Gamma]\ \ \forall (a,b) \in [\lambda]$.[28]

If λ is a Θ-dominant non-polynomial weight put $v(\Theta,\lambda) = 0$.

IV.3.4 Theorem

If $\Theta_1 \subseteq \Theta_2 \subseteq \Omega$ and λ is Θ_2-dominant then

$$v(\Theta_2,\lambda) = \sum_{w \in \{\Theta_1|\Theta_2\}} \operatorname{sgn}(w) v(\Theta_1, w\circ\lambda).$$

Before we come to the proof of IV.3.4 we deduce the promised character formula:

IV.3.5 Corollary

Let λ be a Ψ-dominant weight. For each subset $\Gamma' \subseteq \Psi$ we have

$$[V(\Omega,\Gamma,\lambda)] = \sum_{w \in W_\Psi} \operatorname{sgn}(w) \cdot [S(\Omega,\Gamma')^{w\circ\lambda}].$$

Proof

Take $\Theta_1 = \emptyset$, $\Theta_2 = \Psi$ and $\Gamma = \Gamma'$ in IV.3.4, to get

$$v(\Omega,\Gamma',\Psi,\lambda) = \sum_{w \in W_\Psi} \operatorname{sgn}(w) v(\Omega,\Gamma',\emptyset,w\circ\lambda).$$

Clearly $v(\Omega,\Gamma',\emptyset,\mu) = [S(\Omega,\Gamma')^\mu]$ for any weight μ. It is shown in the proof of III.3.1 that if $\Gamma' \subseteq \Psi$ then $v(\Omega,\Gamma',\Psi,\lambda) = v(\Omega,\emptyset,\Psi,\lambda)$, which is the character of $V(\Omega,\Gamma,\lambda)$ by II.3.1. □

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28 This condition says that if I is the canonical index of weight λ and $\xi = T_I^\lambda$ for $i \in I(a,f)$, then $\xi_{i,j} \in S(\Omega,\Gamma)$. The μ-coefficient of $v(\Theta,\lambda)$ is the number of Θ-standard basis elements $\xi_{i,j} \in S(\Omega,\Gamma)^\lambda$.

Proof of IV.3.4

Let $P(\Theta_1,\Theta_2)$ be the statement that the theorem holds for $\Theta_1\subseteq\Theta_2\subseteq\Omega$ and all Θ_2-dominant weights λ. Suppose that $\Theta_1\subseteq\Theta_2\subseteq\Theta_3\subseteq\Omega$. Using IV.3.1(iii) and IV.3.2 it is easy to check that:

(a) If $P(\Theta_1,\Theta_2)$ holds then $P(\Theta_1,\Theta_3)$ and $P(\Theta_2,\Theta_3)$ are equivalent.

By (a) it suffices to prove $P(\emptyset,\Theta)$ for all $\Theta\subseteq\Omega$.

We will say that disjoint subsets $\Theta_1, \Theta_2\subseteq\Delta$ are ***linked*** if they are linked in the Dynkin diagram of Δ, i.e. if $\exists a\in \underline{n-2}$ such that $\alpha_a\in\Theta_1$, $\alpha_{a+1}\in\Theta_2$ or $\alpha_a\in\Theta_2$, $\alpha_{a+1}\in\Theta_1$; otherwise we will say that they are ***unlinked***. We will say that a subset of Δ is ***connected*** if it is not possible to write it as the disjoint union of two non-empty unlinked subsets. Any subset of Δ can be written in a unique way as a disjoint union of non-empty connected subsets which are pairwise unlinked.

$P(\emptyset,\emptyset)$ is certainly true, so using (a) and induction it is enough to prove $P(\Theta_1\cup\Theta_2, \Theta_1\cup\{\alpha_{u-1}\}\cup\Theta_2)$, where $\Theta_1\cup\{\alpha_{u-1}\}$ and Θ_2 are disjoint and unlinked, Θ_1 is either empty or has the form $\{\alpha_t, \alpha_{t+1}, \cdots, \alpha_{u-2}\}$ for some $t\leq u-2$, and $\Theta_1\cup\{\alpha_{u-1}\}\cup\Theta_2\subseteq\Omega$. Put $\Theta'=\Theta_1\cup\Theta_2$, and $\Theta=\Theta'\cup\{\alpha_{u-1}\}$. The proof of $P(\Theta',\Theta)$ is an elementary if somewhat pernickety piece of combinatorics. The idea is to generalize the map used in the proof of III.2.8. For notational simplicity we will assume that $\Theta_1=\{\alpha_1, \alpha_2, \cdots, \alpha_{u-2}\}$ with $u\geq 2$ (if $u=2$ then $\Theta_1=\emptyset$).

It is routine to check that the distinguished coset representatives of $W_{\Theta'}$ in W_Θ are:

$$1, {}^{s}\alpha_{u-1}, {}^{s}\alpha_{u-1}{}^{s}\alpha_{u-2}, \cdots, {}^{s}\alpha_{u-1}\cdots{}^{s}\alpha_1$$

Let $\lambda\in\Lambda(n,f)$ be Θ-dominant. Put $\lambda(0)=\lambda$, and for $b\in \underline{u-1}$ put

$$\lambda(b) = {}^{s}\alpha_{u-1}{}^{s}\alpha_{u-2}\cdots{}^{s}\alpha_{u-b}\circ\lambda = (u-b\ \ u)\circ\lambda(b-1).$$

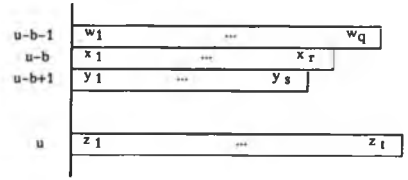
Firstly we deal with the case where some $\lambda(b)$ fails to be polynomial. If $a\in \underline{n}$ we have

$$\lambda(b)_a = \begin{cases} \lambda_a & \text{if } a \in \{1, \cdots, u-b-1\} \cup \{u+1, \cdots, n\} \\ \lambda_{a+1} - 1 & \text{if } a \in \{u-b, \cdots, u-1\} \\ \lambda_{u-b} + b & \text{if } a = u, \end{cases}$$

so if any $\lambda(b)$ is not polynomial, $\lambda_u = 0$ and no $\lambda(b)$ is polynomial for $b \geq 1$. In this case the required formula reduces to $v(\Theta,\lambda) = v(\Theta',\lambda)$, which holds because the u^{th} row of $[\lambda]$ is empty.

Now suppose that $\lambda_u \geq 1$, so that all $\lambda(b)$ are polynomial. For $b \in \{0, 1, \cdots, b-1\}$ let X_b be the set of $\lambda(b)$-tableaux which are row semi-standard, Θ_2-column standard, and adapted to $[\Omega|\Gamma]$. Take $\xi \in X_b$ and write it as follows:

(b) ξ:



Note that the above diagram illustrates the 'generic case' where $1 < b < u-1$. The extreme cases are similar but the diagram has less rows of interest. Here

$$\begin{aligned} q &= \lambda(b)_{u-b-1} = \lambda_{u-b-1} \\ r &= \lambda(b)_{u-b} = \lambda_{u-b+1} - 1 \\ s &= \lambda(b)_{u-b+1} = \lambda_{u-b+2} - 1 \\ t &= \lambda(b)_u = \lambda_{u-b} + b. \end{aligned}$$

Define integers $d_b(\xi)$ and $e_b(\xi)$ as follows:

$$d_b(\xi) = \begin{cases} \max\{c \in \{b+1, b+2, \cdots, b+r+1\} \;/\; \text{if } c > b+1 \text{ then } z_c \geq x_{c-b-1}\} & \text{if } b \geq 1 \\ 1 & \text{if } b = 0. \end{cases}$$

(The range for c is valid since when $b \geq 1$, $b+r+1 = b+\lambda_{u-b+1} \leq b+\lambda_{u-b} = t$, using

the Θ-dominance of λ.)

$$e_b(\xi) = \begin{cases} \max\{c \in \{b, b+1, \dots, t\} \ / \ \text{if } c > b \text{ then } z_c \le w_{c-b}\} & \text{if } b < u-1 \\ 0 & \text{if } b = u-1. \end{cases}$$

(The range for c is valid since $q = \lambda_{u-b-1} \ge \lambda_{u-b} = t-b$.)

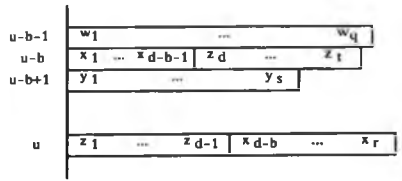
Put

$$X'_b = \{\xi \in X_b \ / \ \xi \text{ is } \Theta'\text{-column standard}\};$$
$$X''_b = \{\xi \in X'_b \ / \ d_b(\xi) \le e_b(\xi)\}.$$

$d_0(\xi)=1$ implies that X''_0 consists of exactly those λ-tableaux which are adapted to $[\Omega|\Gamma]$, row-semi-standard, Θ'-column standard, but not Θ-column standard. $X''_{u-1} = \emptyset$.

If $b \ge 1$ define a map $\psi_b: X_b \to X_{b-1}$ as follows: let $\xi \in X_b$ be as depicted in (b). Put $d = d_b(\xi)$; then $\zeta = \psi_b(\xi)$ is the $\lambda(b-1)$-tableau whose a^{th} row is the a^{th} row of ξ if $a \notin \{u-b, b\}$, and whose u-b and u rows are as in the following diagram:

(c) ζ:



The lengths of the u-b and u rows are respectively

$$(d-b-1) + (t-d+1) = t-b = \lambda(b-1)_{u-b}$$

and

$$(d-1) + (r-d+b+1) = r+b = \lambda(b-1)_u,$$

as they should be. ζ is row semi-standard by definition of d. ψ_b preserves Θ_2-column standardness since $\Theta_1 \cup \{\alpha_{u-1}\}$ and Θ_2 being unlinked implies that neither of

the altered rows is involved in checking Θ_2-column standardness. That ψ_b preserves the property of being adapted to $[\Omega|\Gamma]$ follows from III.2.7 (much as in the proof of III.2.8), using the fact that $u \sim_\Omega u-b$ since $\Theta_1 \cup \{\alpha_{u-1}\}$ is contained in Ω.

We claim that:

(d) $\forall b \in \underline{u-1}$ ψ_b sets up a bijection $X'_b \setminus X''_b \to X''_{b-1}$.

Assuming (d) we can complete the proof. Let v'_b and v''_b denote the elements of $\mathbb{Z}\Lambda(n,f)$ whose μ-coefficients are respectively the numbers of tableaux in X'_b and X''_b with weight μ. Since the map ψ_b is weight-preserving, we see from (d) that for all $b \in \underline{u-1}$, $v'_b = v''_b + v''_{b-1}$. Recalling that $v''_0 = v(\Theta',\lambda) - v(\Theta,\lambda)$ and $v''_{u-1} = 0$, and noting that $v'_b = v(\Theta',\lambda(b))$, we have

$$v(\Theta,\lambda) = v(\Theta',\lambda(0)) - v(\Theta',\lambda(1)) + \cdots + (-1)^{u-1}.v(\Theta',\lambda(u-1)),$$

which is the required result.

It remains to establish (d). Suppose that $1 \le b \le u-1$. Since $\lambda(b)$ is Θ'-dominant, a $\lambda(b)$-tableau is Θ'-column standard iff it is $\{\alpha\}$-column standard for each $\alpha \in \Theta'$.

Step 1: If $\xi \in X'_b$ then $\psi_b(\xi)$ is $\{\alpha\}$-column standard $\forall\, \alpha \in \Theta' \setminus \{\alpha_{u-b-1}\}$.

Here and below we will write ξ as in (b). The only $\alpha \in \Theta' \setminus \{\alpha_{u-b-1}\}$ for which $\psi_b(\xi)$ can fail to be $\{\alpha\}$-column standard is α_{u-b}. If $b = 1$, $\alpha_{u-b} \notin \Theta'$ so there is nothing to do. Otherwise we must check that $z_c < y_{c-b}$ $\forall\, c \in \{d, d+1, \cdots, b+s\}$. The definition of $d = d_b(\xi)$ gives

$$z_c \le z_{c+1} < x_{c-b} \quad \forall\, c \in \{d, d+1, \cdots, b+r\},$$

which is enough, since $s \le r$ and $x_c < y_c$ $\forall\, c \in \underline{s}$.

Step 2: If $\xi \in X_b$ then $\psi_b(\xi)$ is $\{\alpha_{u-b-1}\}$-column standard iff $e_b(\xi) < d_b(\xi)$.

If $b = u-1$ this is vacuous, whilst if $b < u-1$ $\psi_b(\xi)$ is $\{\alpha_{u-b-1}\}$-column standard iff $z_c > w_{c-b}$ $\forall\, c \in \{d, d+1, \cdots, t\}$, and this holds iff $e_b(\xi) < d_b(\xi)$.

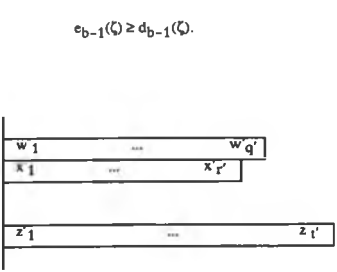
Step 3: $\psi_b(X'_b \backslash X''_b) \subseteq X''_{b-1}$.

Suppose $\xi \in X'_b \backslash X''_b$. By steps 1 and 2, $\zeta = \psi_b(\xi)$ is Θ'-column standard. To show that $\zeta \in X''_{b-1}$ we must verify that

$$e_{b-1}(\zeta) \geq d_{b-1}(\zeta).$$

Write ζ in the form

(e) ζ:



If $d \leq c \leq b+r = t'$ then $z'_c = x_{c-b} > z_{c+1} = w'_{c-(b-1)}$, and so $e_{b-1}(\zeta) < d$. On the other hand, $z'_{d-1} = z_{d-1} \leq z_d = w'_{(d-1)-(b-1)}$, so $e_{b-1}(\zeta) = d-1$. If $b = 1$, $d_{b-1}(\zeta) = 1$, while $d \geq b+1 = 2$ and we are done. Otherwise, if $d \leq c \leq b+s = (b-1)+r'+1$, then $z'_c = x_{c-b} < y_{c-b} = x'_{c-(b-1)-1}$, so $d_{b-1}(\zeta) \leq d-1 = e_{b-1}(\zeta)$, as required.

Step 4: $X''_{b-1} \subseteq \psi_b(X_b)$.

Take $\zeta \in X''_{b-1}$, put $d = e_{b-1}(\zeta)+1 \geq 2$, and write ζ as in (c) and (e). Let ξ be the $\lambda(b)$-tableau in (b). We claim that $\xi \in X_b$ and $\zeta = \psi_b(\xi)$. As usual, ζ being Θ_2-column standard and adapted to $[\Omega|\Gamma]$ implies the same properties for ξ. The definition of $e_{b-1}(\zeta)$ implies that

(f) $$z_c = w'_{c-b} < z'_{c-1} = x_{c-b-1} \quad \forall c \in \{d+1, \ldots, b+r+1\},$$

and so in particular,

(g) $$x_{d-b} > z_{d+1} \geq x_{d-b-1} \quad \text{if } b+r \geq d \geq b+2.$$

Moreover we cannot have $e_{b-1}(\zeta) = b-1$ since $e_{b-1}(\zeta) \geq d_{b-1}(\zeta) \geq b$, so

(h) $$z_d = w'_{d-b} \geq z'_{d-1} = z_{d-1}.$$

Also

(i) $$z_d = w'_{d-b} \geq w'_{d-b-1} = x_{d-b-1} \quad \text{if } d > b+1.$$

Conditions (g) and (h) show that ξ is row semi-standard, and hence in X_b. Conditions (f) and (i) show that $d_b(\xi) = d$, giving $\zeta = \psi_b(\xi)$.

Step 5: Let ξ be as in step 4. Then $\xi \in X'_b \backslash X''_b$.

To check that $\xi \in X'_b$ we must show that ξ is Θ'-column standard. The only $\alpha \in \Theta'$ for which $\{\alpha\}$-column standardness of ξ can fail are α_{u-b-1} and α_{u-b}. For the former, either $b=u-1$ (nothing to do), or $b < u-1$ and

$$x_c = z'_{c+b} > w'_{c+1} = z_{c+b+1} \geq z_{c+b} > w_c \qquad \forall\, c \in \{d-b, \cdots, r\}$$

by definition of $d = e_{b-1}(\zeta)+1$. For the latter, either $b=1$ (nothing to do because $\alpha_{u-1} \notin \Theta'$), or $b > 1$ and

$$x_c = z'_{c+b} < x'_c = y_c \text{ for all } c \in \{d-b, \cdots, s\}$$

since $d_{b-1}(\zeta) \leq e_{b-1}(\zeta) = d-1$. We cannot have $\xi \in X''_b$, for then $\zeta = \psi_b(\xi)$ would not be $\{\alpha_{u-b-1}\}$-column standard by step 2.

We can now finish the proof of (d): steps 3,4 and 5 together show that if $b \in \underline{u-1}$ then

$$\psi_b(X'_b \backslash X''_b) = X''_{b-1}.$$

The injectivity of ψ_b follows from the proof of step 3, which shows that

$e_{b-1}(\psi_b(\xi)) = d-1$, so that we can recover $d_b(\xi)$ and hence ξ itself from $\psi_b(\xi)$. □

§4 (A') when n = 3

Theorem (A') holds when n = 3.

Proof

If either of Ω or Γ is not equal to Δ, we have $|\Psi| \leq 1$, and the situation is covered by III.4.6. Now suppose that $\Omega = \Gamma = \Delta$, and take Ψ-dominant weights $\lambda, \mu \in \Lambda(3,f)$. It is enough to show that $\mathrm{Ext}^i_{S(\Delta,\emptyset)}(V(\Delta,\Delta,\lambda), k(\mu)) = 0$ for all $i > 0$. Let 1 be the canonical index of weight λ, and put $\alpha = \alpha_1$, $\beta = \alpha_2$. By III.3.1 $V(\Delta,\Delta,\lambda)$ has a resolution of the following form:

(a) $$0 \to M_\alpha \cap M_\beta \to M_\alpha \amalg M_\beta \to S(\Delta,\emptyset)^\lambda \to V(\Delta,\Delta,\lambda) \to 0.$$

We will show that (a) is a resolution of $V(\Delta,\Delta,\lambda)$ by $\mathrm{Hom}_{S(\Delta,\emptyset)}(\ ,k(\mu))$-acyclic modules, so that applying $\mathrm{Hom}_{S(\Delta,\emptyset)}(\ ,k(\mu))$ to the deleted form of (a) and taking the i^{th} homology yields $\mathrm{Ext}^i_{S(\Delta,\emptyset)}(V(\Delta,\Delta,\lambda), k(\mu))$ (see for example [Gr; Remark 3, p.148]). Acyclicity is clear for $S(\Delta,\emptyset)^\lambda$, and for M_α and M_β it follows from III.4.3(i). It will follow for $M_\alpha \cap M_\beta$ once we show that

if $\kappa \trianglelefteq \nu$ and ν is a weight of $M_\alpha \cap M_\beta$ then κ is non-dominant,

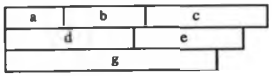
for then $M_\alpha \cap M_\beta$ has an $S(\Delta,\emptyset)$-projective resolution whose terms are direct sums of modules $S(\Delta,\emptyset)^\kappa$ with κ non-dominant.

Let ν be a weight of $M_\alpha \cap M_\beta$, and suppose that $\kappa \trianglelefteq \nu$. By II.3.1 and III.2.8 ν is the left weight of some basis element $\xi_{i,j} \in {}^\nu S(\Delta,\emptyset)^\lambda$ which is neither $\{\alpha\}$-standard nor $\{\beta\}$-standard. We may assume that i is row semi-standard:

$\mathbf{T}_i^\nu$

1	1 ··· 1 | 2 ···· 2 | 3 ···· 3
2	2 ··· 2 | 3 ··· 3
3	3 ··· 3

Denote the sizes of each of the boxes in the above diagram by the letters a, b, c, d, e, g, as in the diagram



The non-standardness conditions imply that $d < g$, and either $a < d$ or $a+b < d+e$. In either case we have

$$\kappa_3 \geq \nu_3 = c+e+g > c+d+e > a = \nu_1 \geq \kappa_1,$$

so κ is not dominant. We have shown that (a) is a resolution of the required type.

Now M_α and M_β are (by definition) generated by weight vectors for non-dominant weights, and we have just demonstrated that no weight of $M_\alpha \cap M_\beta$ is dominant, so applying the functor $\mathrm{Hom}_{S(\Delta,\emptyset)}(\ ,\ k(\mu))$ to the resolution (a) shows that

$$\mathrm{Ext}^i_{S(\Delta,\emptyset)}(V(\Delta,\Delta,\lambda),\ k(\mu)) = 0 \quad \forall\ i > 0,$$

as required. □

Index of Notation

The following is a short list containing some frequently-used notation which is not defined in the main text. (Undefined notation will be assumed to carry its standard meaning.)

We write $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$.
If X is a finite set we write $|X|$ or $\#X$ for the cardinal of X.
If $m \in \mathbb{N}$ we write $\underline{m}$ for the set $\{1, 2, \cdots, m\}$.
$\mathrm{sgn}(\pi)$ denotes the sign of the permutation π.

Unless stated otherwise k denotes a fixed infinite field, dim means dimension over k, and unadorned $\otimes$ denotes tensor product over k.
$\coprod$ denotes coproduct (of modules).

Unless stated otherwise $n, f \in \mathbb{N}$ are fixed natural numbers.

If U is a k-algebra, rad$U$ denotes the Jacobson radical of U; mod$U$ and mod'U denote respectively the categories of (left) U-modules and right U-modules which are finite dimensional over k.

The remainder of this section is a list of notation and terms used in this thesis. Notation which is defined and used only within a single subsection is not usually listed.

Notation	Meaning	Definition
$A = A_f$	the part of $k[M]$ which is homogeneous of degree f	14
$A(X)$	image of $k[X]$ in A_f	16
$A(\Omega,\Gamma)$	$A(P_{\Omega,\Gamma})$	18

Notation	Meaning	Definition
(A)	the statement "the simple $S(B)$-module $k(\lambda)$ is $\mathrm{Ind}_{S(B)}^{S(G)}$-acyclic for all dominant weights λ"	3
(A')	the statement "the simple $S(\Omega,\emptyset)$-module $k(\lambda)$ is $\mathrm{Ind}_{S(\Omega,\emptyset)}^{S(\Omega,\Gamma)}$-acyclic for all Ψ-dominant weights λ"	4
$a(\Omega,\Xi)$, $a(\Gamma,\Xi)$, $a(\Xi)$		41
$a \sim_\Psi b$	a and b belong to the same Ψ-block	10
adapted to $[\Omega\|\Gamma]$		94
anti-dominant		12
B^-, B^+	groups of lower and upper triangular matrices	5
(B)	the statement "the restriction functor mod $S(G) \to$ mod $S(B)$ preserves Ext groups"	3
(B')	the statement "if $\Omega \supseteq \Gamma$ the restriction functor mod $S(\Omega,\Gamma) \to$ mod $S(\Omega,\emptyset)$ preserves Ext groups"	4
$\mathfrak{b}^-$, $\mathfrak{b}^+$	lower and upper triangular subalgebras of $\mathfrak{sl}_n(k)$	83
C_a	a^{th} column	11
$C_{\Xi,a}$	$(\Xi,a)^{th}$ Ψ-column	11
$C(\lambda,\Psi)$	Ψ-column stabilizer of $[\lambda]$	11
$C(\lambda)$	Bernstein-Gelfand-Gelfand resolution	85
(C)	the statement "$\mathrm{Tor}_i^{S(\Omega,\emptyset)}(S(\Omega,\Gamma), V(\Omega,\Gamma,\lambda)) = 0 \ \forall\, i > 0$ and $\mathrm{Ext}^i_{S(\Omega,\emptyset)}(V(\Omega,\Gamma,\lambda), k(\mu)) = 0 \ \forall\, i > 0, \forall$ Ψ-dominant weights μ"	4
$c_{a,b}$	coordinate function on $M_n(k)$	13
$c_{i,j}$	$\prod_{\varphi \in \underline{f}} c_{i\varphi,j\varphi}$	13, 19
c_λ	$c_{I,J}$ where $\lambda = \mathrm{wt}(I)$	14

Notation	Meaning	Definition
$c \circ \xi$	right action of $S(\Omega,\Gamma)$ on $A(\Omega,\Gamma)$	17
canonical index		7
character		91
column		11
contravariant duality		20
contravariant form		20
$D(\Omega,\Gamma,\lambda)$	Schur module of highest weight λ	24
$D'(\Gamma,\Omega,\lambda)$	k-span $\{d_{i,j} \mid i \sim_\Omega i\}$	27
$d_{i,j} = d^{\lambda}_{i,j}$	$\sum_{\pi \in C(\lambda,\Psi)} \operatorname{sgn}(\pi) \cdot c_{i,j\pi}$	26
degree (of a weight)		6
dominant		7
e	evaluation map $M \to S$	17
entry		7
f	a fixed positive integer, the f in A_f	
faulted		53
$G = GL_n(k)$	general linear group	6
$\mathcal{g}$	$sl_n(k)$	83
$\mathcal{h}$	diagonal subalgebra of $sl_n(k)$	83
Ind_U^V	the left exact induction functor modU → modV	48
I(n,X), I(n,f)	sets of indices	7, 8
$i(\Theta)$	canonical index of weight $\lambda(\Theta)$	77
$i \le j$	$i_x \le j_x \ \forall\ x \in X$	7
$i <_l j$		15
$i \sim j$	$j = i\pi$ for some $\pi \in P(X)$	7
$i \sim_l j$		15
$i \sim_\psi j$	$i_x \sim_\psi j_x \ \forall\ x \in X$	10
$(i,j) \sim (i',j')$	$(i',j') = (i,j)\pi$ for some $\pi \in P(X)$	7
index		7
J	transposition anti-automorphism and functor	20
J_a		36
j(a,b)	$\binom{a+b-1}{b}$	40

Notation	Meaning	Definition
$K(P,\lambda)$	$S(P) \underset{U(\delta^-)}{\otimes} C(\lambda)$	86
$K(\Theta,\Theta')$		78
$K(\tau,\sigma)$		68
k	fixed infinite ground field	
k[X]	coordinate ring of X	16
$k(\lambda)$	simple $S(\varnothing,\varnothing)$-module of weight λ	21
	and simple $\hat{n}$-module of weight λ	83
$L(\Omega,\Gamma,\lambda)$	simple $S(\Omega,\Gamma)$-module of highest weight λ	30
$L(\lambda)$	simple $sl_n(k)$-module of highest weight λ	85
l	usually the canonical index of weight λ	
$l(\alpha,t)$		52
l(w)	length of $w \in W$	9
$M = M_n(k)$	monoid of all $n \times n$ matrices	13
$M_\alpha = M_\alpha(\Omega,\Gamma,\lambda)$		53
$M_{\alpha,r} = M_{\alpha,r}(\Omega,\Gamma,\lambda)$		53
$M(\lambda)$	Verma module of highest weight λ	84
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$	
n	a fixed positive integer, the n in GL_n	
n^-, n^+	strictly lower and upper triangular subalgebras of $sl_n(k)$	83
P(X), P(f)	groups of permutations of X and f	7, 8
$P_{i(1),\cdots,i(r)}$		8
$P_{\Omega,\Gamma}$		18
parabolic subgroup of W		9
polynomial weight		6
q	$\max\{\lambda_b, \lambda_{b+1}\} - \lambda_{b+1}$	53
quasi-hereditary algebra		35
R_a	a^{th} row	11
row		11
$R_{a_1,a_2,\cdots,a_r(i(1),i(2),\cdots,i(r))}$		8
$S = S_f$	Schur algebra, the dual algebra of A_f	14
S(X)	dual algebra of A(X)	16
$S(\Omega,\Gamma)$	$S(P_{\Omega,\Gamma})$	18

Notation	Meaning	Definition
$s(I,I')$		68
$s(w,w')$		84
$s(\Sigma,\Sigma')$		79
$\mathcal{sl}_n(k)$	Lie algebra of $n \times n$ matrices over k	83
s_α	simple reflection corresponding to $\alpha \in \Delta$	9
Schur algebra		14
sgn	sign of a permutation	
shape		10
standard (index, tableau)		11
standard (basis element)		28
T	group of diagonal matrices	6
$T = T^\lambda$	basic λ-tableau	10
T_i	the composite map $i \cdot T: [\lambda] \to \underline{f} \to \underline{n}$	10
tableau		10
$U(a)$	universal enveloping algebra of the Lie algebra a	84
V^J	image of V under the transposition functor	20
V^λ, ${}_\lambda V$	right and left λ-weight spaces of V	21
V^o	contravariant dual of V	20
V^*	linear dual of V	
$[V]$	character of V	91
$V(\Omega,\Gamma,\lambda)$	Weyl module of highest weight λ	24
$v(c,\lambda)$		42
$v(\Theta,\lambda) = v(\Omega,\Gamma,\Theta,\lambda)$		92
W	Weyl group	9
W_Ψ	parabolic subgroup of W corresponding to $\Psi \subset \Delta$	9
weight		6
weight of a module		21
weight of a tableau		93
weight space		21
w_Θ		75
$w\lambda$	usual action of $w \in W$ on $\lambda \in \mathbb{R}^n$	9
$w \circ \lambda$	dot action of $w \in W$ on $\lambda \in \mathbb{R}^n$	9
wt(i)	weight of the index i	7

Notation	Meaning	Definition
$\mathbb{Z}\Lambda(n,f)$	free $\mathbb{Z}$-module with basis $\Lambda(n,f)$	91
α_b	the simple root $\varepsilon_a-\varepsilon_{a+1}$	8
$\alpha < \beta$	the natural order on the set of simple roots	75
Γ	a subset of Δ	17
Δ	the set of simple roots	8
∇	comultiplication on k[M]	13
δ	the weight (n-1, n-2, ···, 0)	9
$\delta_{a,b}$	Kronecker delta, equal to 1 if a=b, 0 otherwise	
$\partial(I,I')$		68
ε	counit on k[M]	13
e_a	standard basis vector of $\mathbb{R}^n$	8
θ	canonical map $U(\mathfrak{b}^-) \to S(\mathfrak{B}^-)$	85
$\{\Theta_1\|\Theta_2\}$	set of distinguished coset representatives	92
$\Lambda(n)$	the set of weights	6
$\Lambda(n,f)$	the set of polynomial weights of degree f	7
$\Lambda^+(n,f)$	the set of dominant weights in $\Lambda(n,f)$	42
$\hat{\lambda}$	evaluation on c_λ	21
$\lambda(\Xi)$	Ψ-component of λ corresponding to the Ψ-block Ξ	42
$\lambda(\Theta)$	$w_\Theta \circ \lambda$	77
$\lambda \subseteq \mu$,	subweight partial order	6
$\lambda \trianglelefteq \mu$	dominance partial order	6
$\lambda \trianglelefteq_\Psi \mu$	$\mu-\lambda$ is a sum of elements taken from Ψ	10
$\lambda <_{lex} \mu$	lexicographic order	6
$\|\lambda\|$	degree of λ	6
$[\lambda]$	shape of λ	10
$\xi_{i,j}$	dual basis element to $c_{i,j}$	14
ξ_λ	dual basis element to c_λ	14
$\xi \circ c$	left action of $S(\Omega,\Gamma)$ on $A(\Omega,\Gamma)$	17
$\xi_{\Theta,\Theta'}$	$\xi_{i(\Theta),i(\Theta')}$	77
Φ	the set of roots	8
Φ^+	the set of positive roots	8
Ψ	(usually) $\Omega \cap \Gamma$	25

Notation	Meaning	Definition
Ψ_λ	$\{\alpha \in \Psi \ / \ s_\alpha \circ \lambda$ is polynomial$\}$	74
Ψ-block	W_Ψ-orbit on $\underline{n}$	10
Ψ-column		11
Ψ-component		42
Ψ-dominant		10
Ψ-hook weight		75
$\psi(\Theta,\Theta')$		78
Ω	a subset of Δ	17
$[\Omega\|\Gamma]$		17
$(\ ,\ )$	inner product on $\mathbb{R}^n$	8
$<\ ,\ >$	contravariant forms $S(\Omega,\Gamma) \times A(\Gamma,\Omega) \to k$	21
	and $V(\Omega,\Gamma,\lambda) \times D(\Gamma,\Omega,\lambda) \to k$	24

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